Civic Discourse in Mathematics Classrooms

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Contents

Introduction and Overview 4
A bit more detail 5
Crafting mathematically powerful classrooms: Principles 7
  Overarching principles 7
  Inside the classroom: The essential dimensions of mathematically powerful classrooms 7
Crafting mathematically powerful classrooms: Practices 11
  TRU Dimension #1: Powerful Practices for crafting rich mathematical environments 11
  TRU Dimension #2: Powerful Practices related to supporting sense-making and "productive struggle" 11
  TRU Dimensions #3 and #4: Powerful Practices to Support Every Student's Mathematical Agency, Ownership of ideas, and Identity 12
  TRU Dimension #5: Using Formative Assessment to enhance student learning 14
Expanding the curriculum: A range of issues that can be mathematized 15
Elementary School 20
  Issues and Challenges Particular to Elementary School Mathematics 20
  Three Annotated Elementary School Lessons 21
    Lesson A: Fairness 21
    Lesson B: Climate Justice 24
    Lesson C: Social Justice 26
Middle School 30
  Issues and Challenges Particular to Middle School Mathematics 30
  Three Annotated Middle School Lessons 31
    Lesson A: Fairness 31
    Lesson B: Climate justice 33
    Lesson C: Social Justice 35
Secondary School 38
  Issues and Challenges Particular to Secondary School Mathematics 38
  Three Annotated Secondary School Lessons 39
    Lesson A: Fairness 39
    Lesson B: Climate justice 41
    Lesson C: Social Justice 43
Steps toward implementation, Part 1: Context and Professional development 45

Steps toward implementation, Part 2: Funding 46
Appendix A: The ICUCARE Framework 47
Appendix B: Resources for Supporting Language (including the "Math Language Routines") .. 48
Appendix C: Resources and examples of teacher moves to support discussions 52
Appendix D: Five Practices for Orchestrating Productive Mathematics Discussions 53
References 54

Introduction and Overview

This report is part of a joint project by the National Academy of Education and the National Council of Teachers of Mathematics. The project aims to support civic reasoning and discourse in the schools, helping teachers across disciplines to prepare youth in the United States to examine and discuss complex issues in ways that are thoughtful, developmentally appropriate, and respectful of others' opinions and experiences. In mathematics, this can mean mathematizing issues that are meaningful and relevant to students' lives – issues that may be related to current curricular topics (e.g., aspects of "fairness") or that are not dealt with in many curricula (e.g., making decisions regarding safe social practices during a pandemic or addressing some of the issues involved in voting rights). Addressing potentially controversial topics such as these calls for conceptualizing and representing relevant phenomena, making clear claims, specifying assumptions, supporting justifications with evidence, weighing multiple points of view, and then working through such assumptions, claims, and rationales with others to construct and critique viable arguments.

The curriculum materials that have historically dominated the mathematics textbook market in the U.S. have provided little support for dealing with current or controversial issues, either in terms of content or pedagogy. This report offers such support. It provides background information regarding teaching and learning mathematics, which undergirds powerful classrooms – classrooms from which students emerge as agentive and mathematically, epistemologically, and socially powerful thinkers. It offers a set of principles for powerful pedagogy and practice in mathematics, describing what we can offer students to ensure that every student is enfranchised and supported equitably in thinking, learning, and acting in the world. It briefly describes three curricular strands connected to current or controversial issues (Fairness, Environmental Justice, and Social Justice) that can be addressed with increasing developmental, social, and mathematical sophistication through the grades. It identifies specific developmental considerations for students at the elementary, middle, and secondary grade levels. It then focuses on specifics related to students' mathematical thinking, development, and learning, along with annotated curricular examples for each of the three strands.

To achieve the vision in this document will require facing ongoing challenges regarding curricula and teacher support. Traditional curricula have focused on abstract mathematical ideas, with scant attention to the kind of sense-making that is internal to mathematics (seeking and exploiting mathematical coherence) or that is external to it (making use of real-world phenomena to give meaning to mathematical objects and using mathematics to model such phenomena in meaningful ways), or personal (connecting to previous experiences, understanding, and knowledge), or social (connecting to the collective sense making in a particular classroom).

Within mathematics, for example, students should know that if $3 + 4 = 7$, then they can "undo" the addition and see that $3 = 7 – 4$ and $4 = 7 – 3$; likewise, that doubling can be undone by dividing by 2. They should develop the kind of number sense that tells them that the product of 26 and 39 should be close to 1000, because the product of 26 and 39 should be close to the product of 25 and 40, which equals 1000. They should know how things grow, so that they can say that the perimeter of a circle (or a pizza that the circle represents) is doubled when the radius is doubled, but the area (or total amount of pizza) is quadrupled. And, they should be able to explain their reasoning, which an important aspect of sense-making. In high school, students should be able to explain why the graphs of $y = 2x + 5$ and $y = 3x – 2$ *must* cross, without doing the computation.

External to mathematics, students should learn to make meaningful links between real-world phenomena and their mathematical representations. They should understand, for example, how two graphs represent the costs of two cell phone plans and what the implications are for decision making; what a change in the minimum wage means in terms of purchasing power over the course of a year, or how interest rates on various purchases mount up. Similarly, they should understand how to see the implications of graphs and other mathematical representations that explore the relationships between, for example, real estate values and social services.

Students will need help in learning to do this kind of mathematical sense-making, and teachers will need support in crafting the kinds of classrooms that help students – often collaboratively – engage in these kinds of activities.

The final section of the report addresses next steps. It offers suggestions for professional development to help make these ideas a reality, and thoughts about possible research and development.

A bit more detail

In contrast to science, where new ideas (for example, recombinant DNA and clustered regularly interspaced short palindromic repeats [CRISPR] technologies) quickly enter the curriculum, the mathematics curriculum has been static for decades. Consequently, students often find mathematics less interesting and relevant to their lives than it could be. We hope to support and encourage increased curricular opportunities for student engagement with meaningful and mathematically rich tasks and contexts. The focus of this report is on the unique challenges of providing opportunities for, and supporting, civil reasoning and discourse in K-12 mathematics classrooms.

Essentially, meaningful mathematics learning needs…

(1) Examples in the curriculum to be relevant to students. Although mathematics curricula include some “real world” examples to exemplify applications, these examples are typically simplified to the point where they make little meaningful connections to the phenomena that people experience outside the classroom.
(2) Less emphasis on “answer getting.” Classroom reasoning often focuses on “answer-getting” using demonstrated procedures rather than carefully thinking through the connections underlying situations.
(3) More opportunities for multi-modal communication. There is room for much more classroom discourse and communication of all types, including talking, listening, reading, and writing. Students should have multiple and frequent opportunities to draft and revise their communications. This is especially the case when developing meaningful models of real world phenomena. Doing so calls for deciding the relevance of various issues to the

question at hand, understanding the possibility of more than one point of view, and carefully sorting through alternative approaches to problems.

Addressing these goals will *enrich* mathematics curriculum and instruction.

Mathematics curricula in the US have evolved slowly over the 20^{th} and 21^{st} centuries, but for the most part they have remained true to their foundations. Instruction at the elementary level has historically focused on the mastery of basic operations, measurement, and the properties of geometric objects. (One superintendent's instructions to teachers in the 1890s said that mathematics instruction should cover no more than the addition, subtraction, division, and multiplication of whole numbers, so that students would be prepared to serve as store clerks!) In recent years the elementary curriculum has opened up to include a greater emphasis on the conceptual underpinnings of fractions and ratios, some descriptive statistics and discussions of statistical distributions. Similarly, curricula at the middle and secondary levels have changed some over the past 50 years, both with the inclusion of statistics and the condensation of courses to allow students to study calculus in high school rather than in college. Yet, the standard course sequences and content remain largely the same: preparation for Algebra is followed by Algebra I, Geometry, Algebra II, Pre-calculus, and then Calculus. Examples tend not to be meaningful in terms of students' personal lives, and instruction often focuses on implementing procedures more than individual or collective sense-making.

Much of the power of mathematics resides in its ability to model meaningful real-world phenomena, enabling students to document and discuss (for example) issues of social justice, environmental justice, and fairness. Yet, curricula have typically skirted such issues (Burkhardt & Schoenfeld 2022) and "applications" are often obviously contrived. The result is that many students see little of relevance or interest in mathematics, and others are alienated from it (Gutierrez 2013). Moreover, this kind of curriculum results in learners believing that doing mathematics involves producing the answers to exercises or problems in the textbook rather than developing tools for understanding, describing, and changing the world around them.

There are significant opportunities to open up the mathematics curriculum, at all grade levels, to include tasks that address a broad range of concerns in ways that call for sensemaking and mathematizing, that relate meaningfully to students' lives, that use mathematically reasonable representations of the issues involved, and that helps learners understand how they might use mathematics to improve the world around them. Similarly, it would be helpful to put norms and practices in place that make the mathematics classroom a more inviting place in which students feel comfortable exploring and communicating ideas. This means making it safe for students to take risks as they work through ideas. Doing so has the potential to re-frame mathematics as a domain that (while still beautiful and powerful) can be seen as personally meaningful, purposeful, and accessible. It would make mathematics instruction much more consonant with the larger project of civic discourse (see, e.g., Lee, White, & Dong [2021]), in which students learn, across the curriculum, to address meaningful challenges in thoughtful, analytic, and respectful ways.

Crafting mathematically powerful classrooms: Principles

This section begins with some general principles about learning, reflecting the breadth of coverage in the introduction. We then transition to the mathematics classroom, reflecting on how these broad principles need to be considered when teaching. We then describe the principles underlying powerful mathematics learning classrooms – classrooms from which all students emerge as powerful and empowered mathematical thinkers and learners.

Overarching principles

Human learning is universal and follows certain general patterns (see, e.g., the *Handbook of the Cultural Foundations of Learning* [Nasir et al., 2020]). Key ideas in the *Handbook* are that learning is rooted in the evolutionary, biological, and neurological systems of our bodies and minds; that it is inseparable from our social and cultural activities; that it is not a separate activity, but is intimately tied to development, emotion, and the formation of identity; and that it is inherently a social activity, shaped by everyday life cultural activities, both in and out of school and across the lifespan (see, e.g., the discussion of the RISE framework in Nasir et al. [2021]). These ideas apply to all learning, including the tacit reference to communication in the claim that learning is inseparable from our social and cultural activities. It is essential to keep them in mind, even when we think about what takes place in classrooms.

The goal is for learning environments to be consonant with the general principles discussed above (Seda & Brown, 2021). Hence classrooms should function as learning communities in which everyone contributes to knowledge generation, not just the teacher. Additionally, such learning communities depend on and draw from knowledge outside the classroom. They build critical awareness and reflection, and they support learners' active thinking and problem solving. Students become independent thinkers in these environments through culturally relevant/responsive curricula which builds on students' assets – languages, cultural practices, identities, and prior knowledge and experiences (see Appendix A).

Inside the classroom: The essential dimensions of mathematically powerful classrooms

A fundamental goal of mathematics instruction is to provide the kind of support that enables every student to emerge from mathematics instruction as an agentive and mathematically, epistemologically, and socially powerful thinker. Thus, issues of how students' mathematical identities are developed, manifested, and supported are central.

Literature reveals multiple aspects of identity and its shaping in social contexts (see, e.g., Greeno, 2006; Holland et al., 1998; Koole 2003; Turner et al. 2013; Wenger, 1998). First, identities are shaped in multiple ways – by membership in communities ("I am a reader; I am a member of a great book club"), by stereotypes (e.g., certain groups are considered good at math; certain other groups are not); by the ways people position each other (either explicitly, e.g., "Let's ask José. I bet he knows" or tacitly, when a group ignores a contribution made by one student but listens when another, higher status student makes the same comment)..

Identity is closely related to agency, a willingness to take on tasks and contribute, either individually or as part of a group effort (see Aguirre, Mayfield-Ingram & Martin, 2013). When students actively engage in and contribute to mathematical (or other) discussions, it can be inferred that they have productive mathematical (or other) identities. Students with *conceptual* agency want to understand and make connections (Cobb et al. 2009). Further, *critical mathematical agency* (Turner, 2003) expands on the notion of agency to include the idea of engaging in mathematical activities that are both personally and socially meaningful. This is a key goal of instruction that helps to prepare students for engagement in civic discourse.

As we have noted, agency and identity are constructed through social interactions, including classrooms. In classrooms that provide *all* students with opportunities to explain and justify solution strategies, pose questions, and articulate connections between mathematical ideas, in ways that are personally safe, students have opportunities to become (more) mathematically agentive and build (more) productive mathematical identities.

Mathematical agency and positive mathematical identities are goals for *every* student – this is a fundamental issue of equity. The question is how to get there. That is, how can the important aspects of learning environments be organized so that they can be understood and worked on, for purposes of continuous improvement? The Teaching for Robust Understanding (TRU) Framework (https://truframework.org/; see Schoenfeld 2013, 2014, 2022, Schoenfeld et al, 2023a, 2023b) organizes and distills the large literature on teaching into five essential dimensions of classroom practice (see Figure 1).

Figure 1: The five dimensions of mathematically powerful classrooms

The Five Dimensions of Powerful Classrooms				
The Discipline	**Cognitive Demand**	**Equitable Access to Content**	**Agency, Ownership, and Identity**	**Formative Assessment**
The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers. Discussions are focused and coherent, providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections, and develop productive disciplinary habits of mind.	*The extent to which students have opportunities to grapple with and make sense of important disciplinary ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conducive to what has been called "productive struggle."*	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core disciplinary content being addressed by the class. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.*	*The extent to which students are provided opportunities to "walk the walk and talk the talk" – to contribute to conversations about disciplinary ideas, to build on others' ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.*	*The extent to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understandings.*

This research leads to a series of questions that teachers can ask during planning, teaching, and on reflection:

- What mathematics should these students (at a particular age, given developmental considerations, etc.) experience, and in what ways? (Dimension 1)
- How can instruction be adjusted in ways that position these students so that they can build on prior knowledge and expand knowledge base with new ideas? (Dimension 2)
- How can instruction be organized so that every student in a classroom/school/district participates meaningfully with regard to the content, engaging in relevant mathematical practices, e.g., those in the NCTM and Common Core State Standards? (Dimension 3)
- How does the content relate to students and their personal lives; how does their engagement in classroom discussions provide opportunities for them to get their ideas heard and discussed, in ways that are respectful and enfranchising? (Dimension 4; note the essential connection to civic discourse)
- How is student thinking made available for collective discourse and productive feedback, whether from the teacher or other students? (Dimension 5)

Before expanding on the TRU framework, the next section addresses the central role that language plays in supporting powerful and empowering classrooms. Then the report will offer a collection of practices, organized by dimension, that helps teachers craft mathematically powerful classrooms.

Attending to Language

Using oral or written language is central in each of the 5 dimensions listed above. Teachers bring to the classroom their own meanings for the language they use; they also bring ways of talking that fit outside the classroom (i.e., slang, jokes, how we present evidence). In the classroom, teachers use mathematical terms and phrases with very specific meanings and mathematical ways of talking or writing (i.e., conjecturing, justifying, generalizing, referring to visuals, etc.). Ultimately, we would like students to learn how to think and communicate using those terms and phrases with mathematical meanings and learn to use those valued ways of communicating—in another word, "the mathematical register." The question is how to bridge the gap by building on the everyday register and students' own sense making processes (Moschkovich, 2012 & 2014).

The main language issues in teaching mathematics are hearing the math in student contributions, even when the ways the math is expressed may be imperfect or imprecise; supporting students in communicating mathematical ideas through language using informal and formal registers; and connecting students' ways of talking to more formal registers. Learning to communicate using those informal (everyday, home, etc.) and formal (school, mathematical, or academic) registers is part of dimension 1 in the TRU framework—the discipline. Attention to mathematical language and its relation to everyday language use is essential. At least as important, sensemaking depends on having ideas within reach (dimension 2: cognitive demand) and opportunities to express initial or exploratory ideas through emergent language which is often imperfect (Moschkovich, 2014 & 2016). Sense making is an activity that fundamentally involves language: language is not only a communication tool, it is also a thinking tool. If students are not supported in participating in mathematical conversations and written communication, they don't have equitable access to core math ideas and practices (Dimension 3), and they certainly don't have opportunities to engage in ways that build mathematical identities (Dimension 4). Finally, formative assessment (Dimension 5) fundamentally involves talk, text, and other modes of communication (drawings, graphs, tables, visuals, etc.).

Math understanding and language development are interconnected and mutually reinforcing. Instruction that attends to language needs to focus on *what students are using language for* rather than the exact way that students say, record, or write their contributions. That means doing more than focusing on vocabulary or simply providing students with the mathematical meaning of terms to use or memorize (such as vocabulary lists, definitions, word walls, etc.). Students need to actively use language to develop and negotiate meanings, not only for single words, phrases, or definitions, but also to learn how to express the math practices through language (for example, by learning how to construct a good justification, make a claim more precise, or connect a conjecture to a mathematical representation). As students share their thinking with peers, they can start to refine their language; over time, teachers can support students in moving toward more effective mathematical communication as they revise and refine oral and written expressions of their math work[1].

Language matters in math classrooms for all students: no student walks into a math classroom proficient in formal or academic mathematical communication, and all students need opportunities to use both informal and formal mathematical language to communicate their ideas as they explore a topic or solution. Thus, many of the teaching strategies that support discussions are useful for attending to language.

We recommend using "The 5 Practices for Orchestrating Productive Mathematical Discussions to support discussions"[2]: Anticipate, Monitor, Select, Sequence, and Connect (see Smith and Stein, 2011). In a problem-based curriculum, many activities can be described as "do math and talk about it," but the 5 Practices lend more structure to these activities to help students make connections and learn new mathematics.

We also recommend using the eight Math Language Routines (Zwiers et al, 2017[3]), which provide opportunities over extended time for students to actively participate in and improve using language to communicate mathematically, both orally and in writing. Detailed descriptions of Math Language Routines are in Appendix B. Annotations we provide for each sample lesson in the sections "Three Annotated Elementary/Middle/Secondary School Lessons" include suggestions for using these Math Language Routines.

[1] For an example of such scaffolding during a classroom discussion, see Moschkovich, 2015.

[2] Other useful materials for practitioners include "Classroom discussions: Using math talk to help students learn," by Chapin, O'Connor, & Anderson (2003) and "Intentional talk: How to structure and lead productive mathematical discussions," by Kazemi & Hintz (2014).

[3] For short descriptions of the MLRs see Appendix B; for more detailed descriptions see Daro (2021) and Zwiers et al (2017); for examples of teachers using them in a multilingual secondary classroom see Zahner et al (2021).

Crafting mathematically powerful classrooms: Practices

The previous section outlined five fundamental dimensions of mathematics classrooms. There is compelling evidence that, to the degree that classrooms do well along those five dimensions, the students in them will develop into motivated, agentive, and powerful mathematical thinkers. In this section, we turn from principles to teaching practices – the actions teachers can take that help to move their classrooms in those directions. For each TRU dimension we list a set of classroom practices that help to enrich classroom activities related to that dimension.

TRU Dimension #1: Powerful Practices for crafting rich mathematical environments

The goal of Dimension 1, "The Discipline", is for the mathematics learning to be a rich learning experience. In addition to being focused on big mathematical ideas, students should have opportunities to engage in key mathematical practices including sense-making, building connections, reasoning, and explaining.

To achieve this goal, text materials can be enhanced in a variety of ways. A key idea is to keep students focused on sense-making. This can be done by posing questions for small group discussion as well as in whole-class discussions, such as:

- "Does anyone have ideas about how we can get started?" (This and similar questions can open up discussions)
- "Does the answer seem reasonable?" (If someone's grandmother turns out to be 7 years old, or someone is running at 2 miles per hour, there's a problem!)
- "Does anyone have a different way of thinking about this?" (Such questions can support students in comparing methods and help them make connections, prompt them to share their thinking, and understand each other's way of thinking.)
- "How would you explain this idea (or problem or solution) to a student who missed today's class?" (This makes their peers the audience for their communication.)

Mathematically rich problems can stimulate especially meaningful discussions. First, discussion of "group-worthy" problems – problems that can be approached and solved in more than one way – encourages sense-making and making connections between the insights and methods that students develop (see, e.g., Cohen & Lotan 1997, 2014; Lotan, 2003). For a collection of 100 lessons that include mathematically rich tasks and extended lesson plans to scaffold productive classroom dialogue, see the formative assessment lessons at https://www.map.mathshell.org/lessons.php.

TRU Dimension #2: Powerful Practices related to supporting sense-making and "productive struggle"

The key concepts related to Dimension 2, "Cognitive Demand", are sense-making and productive struggle. Students should actively build their own understandings as they work. Tasks that are too easy don't help them to grow mathematically, but tasks that are inaccessible don't help either – they just build frustration. Therefore, a significant amount of monitoring is called for (see also Dimension 5, "Formative Assessment"), as is making adjustments (i.e., scaffolding) so that students are working on tasks with balanced difficulty levels.

As noted above, using mathematically rich tasks – tasks with multiple entry points, and multiple solutions – allows more ways for students to get into a task, and then to compare and contrast their thinking with others. Teachers cannot adjust tasks or make suggestions productively unless student thinking is observable; this is another reason for collaborative group work, whole class discussions, public records, and student writing. Also, it is difficult for students to revise their thinking unless it is recorded in some way.

Equally important is the construction of classroom norms in which it is fine to make mistakes, reflect on them, and then make progress; solutions to complex problems don't come "all at once." Some of the routines that attend to both mathematics and language include "Three reads," "Stronger and clearer each time" and "Critique, correct, and clarify" in Appendix B.

TRU Dimensions #3 and #4: Powerful Practices to Support Every Student's Mathematical Agency, Ownership of ideas, and Identity

The goal for mathematics instruction is that every student become mathematically powerful, developing a sense of mathematical agency and a positive mathematical identity. For that reason we discuss teaching practices related to Dimensions 3 ("Equitable Access to Content") and 4 ("Agency, Ownership, and Identity") of the TRU Framework together. Equitable access to central mathematical content and practices is crucial – *and*, the right kind of access should contribute to the development of positive mathematics identities.

The key issue with regard to equitable access is to make sure that *every student engages with the core mathematical content of the lesson*, whether in whole class or small group. It is easy for some students to take over and let others ride on their coattails, or for some students to hide. Maintaining a classroom where all students feel free and encouraged to participate, on matters of mathematical import, can be challenging. Not all students will participate in the same way: for example, some students may choose writing over talking, others may talk more in small groups than in whole class discussions. But, for example, having some students practice "basics" while others engage in meaningful problem solving ultimately winds up creating inequities and increasing the performance differences between the two groups. Students are not likely to learn if they do not have the opportunity to engage in problem solving, reasoning, or other key math practices.

Some equity-focused practices include emphasizing sense-making over speed, discussing and highlighting various ways of being mathematically smart (generating ideas, organizing a group's work, serving as sounding board and questioner, recording and summarizing a group's ideas, etc.), and highlighting and building on student contributions so that reluctant sharers see their ideas as welcome. This too takes practice. Instead of looking for and certifying the "right" method for approaching a particular problem, teachers look for the productive elements in student thinking, with the goal of working them into classroom conversations. Student contributions might include ideas, connections, conjectures, explanations, etc. Teachers can "revoice" student contributions, perhaps clarifying or paraphrasing, but giving the students credit for the ideas.

Some teachers use randomizing devices such as equity sticks. When used in non-threatening ways (e.g., so that opting out can be done safely), such randomizing devices can create expectations of participation, distributing opportunities to participate widely. Similarly, students can be invited to use a wide range of resources that support mathematical thinking: everyday and home language, culturally grounded or invented algorithms (which can be compared to standard algorithms), and a wide range of representations including drawings, gestures, objects, and symbols. Table 1 includes a collection of powerful identity-related practices (Aguirre, Mayfield-Ingram, & Martin, 2013).

Table 1: Five Equity Practices

Practice	General Actions	Specific Practices
Go deep with mathematics	Implement tasks that promote reasoning	• Justify claims • High cognitive demand tasks • Multiple solution strategies and representations
Leverage multiple mathematics competencies	Structure collaboration to use a variety of skills, knowledge, all levels of confidence	• Students contribute in different viable ways • Tasks with multiple ways to enter • Multiple ways to communicate • Students with different levels of confidence, skills, and knowledge can engage
Affirm mathematics identities	Support productive struggle ….	• Promote student persistence and reasoning • Students see themselves as making valuable contributions • Assumes mistakes are learning opportunities • Validate students as math learners • Various ways to show competence • Focus on competencies
Challenge spaces of marginality	Position students as sources of expertise for solving complex problems	• Position as sources of expertise • Generates questions • Mathematics authority distributes between students, teachers, and mathematics • Student-to-student interaction • Reasoning is public
Draw on multiple sources of knowledge	Use and connect mathematical representations Elicit and use evidence of students' thinking	• Makes intentional connections • Bridges previous learning to new learning • Taps mathematical knowledge from community, family, culture, etc. • Recognizes and strengthens multiple language forms • Affirms and supports multilingualism

Source: Aguirre, Mayfield-Ingram, & Martin (2013), p. 16

TRU Dimension #5: Using Formative Assessment to enhance student learning
The key to formative assessment is gathering information about student understandings *prior to and during instruction* that can be used to adjust instruction so that it meets the students where they are – for example, supporting rich mathematics, identifying false starts and misconceptions (and dealing with them carefully, not simply making corrections!), making sure the level of cognitive demand is right, and making sure students have meaningful opportunities to have their voices heard.

Giving pre-assessments (assignments that reveal what students know and don't know, before teachers dive deeper into a unit or lesson) allows teachers to learn the knowledge, skills, dispositions students already have and be able to build on them. Similarly, using tasks that reveal misunderstandings, and having classroom discussions of them, allows students to work through their ideas in ways that are more powerful than simply saying what's right or wrong.

Providing students with multiple ways to show their thinking–not only orally, but also in writing (which allows revising); not only in whole class or individually, but also in small groups–supports equitable access to feedback from multiple sources. It is worth noting that having students collaborate makes life *easier* for teachers. When the burden of diagnosis and remediation all falls on the teacher's shoulders, formative assessment is a nearly impossible challenge. But, when students work according to the norms discussed above (asking questions, asking for justifications, etc.), many learning issues get resolved in small groups or pairs, before the teacher even learns about them.

The 100 formative assessment lessons at https://www.map.mathshell.org/lessons.php provide a large number of classroom activities and support materials for middle and secondary school mathematics formative assessment, including extensive lesson plans and examples of supportive discourse practices.

Expanding the curriculum: A range of issues that can be mathematized

Current mathematics curricula tend not to provide students with significant opportunities for modeling, decision-making, and the kinds of civic discourse involved in addressing complex and relevant "real world" problems. The question is: what bodies of work, or approaches, might be useful for students to experience?

To indicate that there is a variety of possibilities, we begin by describing a range of topics that might be explored. Then we zero in on three themes that have two significant properties: (1) in addition to being mathematically rich, these topics have the potential to be personally meaningful for students; and (2) the topics can be approached in various ways, depending in part on what aspects of the topics consider to be important. Thus, discussions of these topics can provide rich opportunities for learning to engage in civic discourse, using mathematics as a tool for reasoning and argumentation.

Among the topics that have the potential to offer mathematical richness and support civic discourse are:

- *Building and discussing models of a wide range of phenomena related to COVID-19 and other infectious diseases* (or, more generally, addressing the mathematics underlying numerous issues of public health). Policies can seem arbitrary, and they may go unquestioned or ignored – but underlying data and policy questions can be reasoned through. Consider issues of vaccination, for example, teachers can ask guiding questions such as what data would you want to gather to see if vaccines have an impact; how would you analyze those data; what is the downside risk of vaccination; what data would you want to gather to explore the risks of vaccination; and how would you analyze those data? One might consider questions of masking, social distancing, and more, in the same ways. There are any number of social issues related to COVID-19 – for example, how do COVID rates correspond to race or ethnicity, to socioeconomic status?

- *Critical literacy skills, specifically the challenges of selective data analysis*. How do you take any current issue that has mathematical underpinnings and make sense of it? This includes critically reading, interpreting, and analyzing data in a situation to figure out what information is needed, what claims can be made, what makes a claim clear and precise, what are sources of strong evidence, etc. At more advanced levels, materials exist for addressing some such issues; see, e.g., Burkhardt & Schoenfeld (2022). As the phrase "lies, damned lies, and statistics" indicates, it is essential for students to learn how to understand what it means to draw fair conclusions from a body of data and how data can be misrepresented. This may sound very advanced for elementary school students. However, these points can be illustrated with relevant examples – e.g., giving one student a large cookie and one student a small cookie and claiming that the distribution is fair because each student got the same number of cookies. (Or, giving each family the same amount of money for holiday gifts, when some families have one child and others have more.)

- *Census–related issues.* Population Reference Bureau (PRB.org) put together a World Population Data Sheet focusing on the census throughout history. "PRB informs people around the world about population, health, and the environment, and empowers them to use that information to advance the well-being of current and future generations." At minimum, students can see videos of issues related to population growth and resource scarcity, and learn to represent the data mathematically.

- *Voting, gerrymandering, etc.* You can find many pointers to lessons if you google "gerrymandering curriculum tasks;" there are games and other tools if you google "gerrymander simulations" or "gerrymandering example." There is a gerrymandering task in Berry et al.'s (2020) book, High School Mathematics Lessons to Explore, Understand, and Respond to Social Injustice. The first example of "fairness" below can be modified for the elementary classroom, where x's and y's can be boys and girls.

There are many possible ways to open up the mathematics curriculum to include topics that are mathematically rich and that have the potential to support rich civic discourse. In what follows, we focus on three such topics and exemplify them with sample lessons.

Three key themes: Fairness, Environmental Justice, and Social Justice

In this section, we briefly describe topics or contexts under each of the three themes of fairness, environmental justice, and social justice that can be used to engage students with rich mathematical content and to support thoughtful and respectful civic discourse across the K-12 spectrum (and beyond). In the next section we provide and annotate one example from each topic at each elementary, middle, and secondary school level, suggesting how the tasks can be considered and used in the classroom.

We stress here that the role of mathematics in all of the arenas discussed below is to help students understand complex situations in which there are often strong opinions that are only partly supported by evidence. Ultimately, students should develop a set of dispositions and skills that enable them to think carefully about important social issues that can be illuminated by mathematics –for example, if they see claims in the media about some form of inequity, they can frame inquiries in neutral ways, decide which factors or variables are relevant, consider what data would inform the issue, and find and analyze the data.

Addressing potentially controversial issues calls for empathy and sensitivity in both personal and interpersonal terms. Some discussions may be potentially painful for some students – for example, a discussion of COVID-related issues may bring to mind the pain and suffering of relatives who contracted the infection and may have died from it. Other topics may raise issues of personal identity or group affiliation. There is always the possibility of essentializing, stereotyping, and other challenging conversational moves that are typically not seen in classrooms that focus narrowly on traditional mathematical topics. Moreover, the background knowledge required for addressing such issues is broader than the knowledge required for teaching straightforward mathematics topics.

For that reason each of the sample tasks described in the next section will include brief annotations related to:

- Knowledge: the underlying knowledge required to address the issue (at this grade level).
- Epistemology: the broad set of issues involved, how the issue is framed, what assumptions are made, and so on.
- Dispositions: the willingness to approach challenging issues while putting preconceptions aside, to weigh evidence fairly, to engage in respectful discourse about sensitive topics, etc.
- Ethics: the recognition that some alternative considerations and possible approaches to situations are based on values, and that there is not necessarily one "right" answer to complex situations.
- Language: the understanding that talk, text, and visual displays are central aspects of learning to reason mathematically, and that the linguistic resources students bring to the classroom can serve as a basis for developing and refining their understandings regarding mathematical situations and communicating about them (see Appendix B).

There are delicate political context issues to consider when addressing issues such as fairness, environmental justice, and social justice in the classroom. Indeed, in states that have banned the teaching of topics such as the enduring legacies of slavery, the very mention of justice may be seen as problematic. Consequently, some teachers or districts may decide that one or more topics we have chosen to exemplify are too hot to handle in the classroom. Our stance is that relevant resources must be widely available, whether or not a particular school or district will choose to use them. One reason we highlighted specific alternatives (Building and discussing models of a wide range of phenomena related to COVID-19 and other infectious diseases; critical literacy skills, specifically the challenges of selective data analysis; Census-related issues; and, Voting, gerrymandering, etc.) at the beginning of this section is to highlight the fact that that schools and districts have many topics to choose from when they think about opening up mathematics lessons to modeling issues suitable for civic discourse. The discussions below indicate the ways in which all such topics can be approached.

Here are brief descriptions of the three key topics.

A. Fairness as a context for mathematics learning

Fairness is a well-studied area of the curriculum. Issues of fairness undergird much of the curriculum at all grade levels. For example, fractions represent "fair shares" of some quantities divided equally among a group of people. How do you share two differently sized cookies fairly between two students? Or a snack-size box of raisins? Things can get complex rapidly. Suppose four people want to share three cookies, and the cookies are not the same size. What about cutting a square frosted cake so that three people each get the same amount of cake and the same amount of frosting?

Or, suppose students want to share the costs of carpooling to school, but each participant lives a different distance from school. How do you think about allocating costs in that case? If the students were riding a bus, they would all pay the same fare, but in some transit systems they would be paying more for longer trips. (For a unit that explores this issue, see https://www.map.mathshell.org/lessons.php?unit=6200&collection=8.) Then there are issues of fairness in representation and apportionment. In Figure 3A, where there are 5 voting districts, Y's are 40% of the population and win 2 out of 5, or 40%, of the five vertical districts. In Figure 3B, Y's are 40% of the population and win of 0% of the five horizontal districts. What's "right" isn't so simple any more.

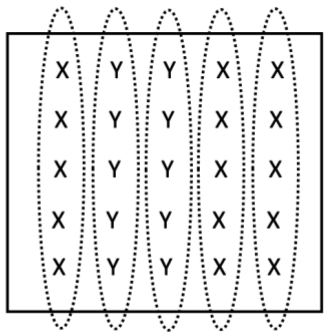

Figure 3A. The Y's win 40% of the districts

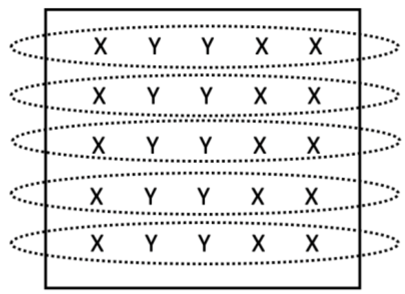

Figure 3B. The Y's lose In every district

Issues of representation and apportionment in the real world are much more complex, as well as highly controversial. Further, while mathematical tools provide one way of determining fairness when it is agreed that "equal shares" or some other form of "equal" is what is considered collectively to be fair, there are also perspectives on what counts as fair that do not rest on notions of equality. For example, if the goal is for every student to have a clear view of the stage during a performance, some students may need extra pillows. The unequal distribution of pillows turns out to be fair!

A collection of curricular resources can be found at https://www.nctm.org/Search/?query=fairness#?wst=d96f7c2823ec8a70319bf3c5b2c7fcd5. That's the tip of the iceberg. Googling individual topics yields a wide range of resources, including voting and redistricting simulations.

B. Environmental Justice as a context for mathematics learning

There is an increasing frequency of major climate disasters. "Once in a century" firestorms now rage on a regular basis; there are torrential rains and flooding across the globe, rising sea levels threaten whole cities and coastlines; and increasing temperatures are making some areas unlivable. At the same time rain forests are being burned to the ground, laws or policies have been enacted that appear to favor one part of the energy market over another, e.g., gas-guzzling trucks are exempted from regulations. Environmental justice is also an urban issue. Environmental injustice results from the interaction of economic disparity, social inequity, and unequal access to environmental or health resources. Many students from communities of color and low-income groups reside and study within expansive urban districts, where they bear the brunt of environmental injustices in glaring disproportion. There is rich source material for assignments and projects at every grade level. (For two examples at the time this report is being compiled, see https://www.nytimes.com/2022/03/09/climate/redlining-racism-air-pollution.html and https://www.washingtonpost.com/climate-environment/2022/03/09/redlining-pollution-

environmental-justice/). Very broad coverage is given in publications such as https://rethinkingschools.org/books/a-people-s-curriculum-for-the-earth/
There is an emerging body of curricular materials related to environmental justice. Many lessons are found in broader collections related to social justice. NCTM resources can be found at https://www.nctm.org/Search/?query=environmental%20justice#?wst=d8a3428bd9d5122a0589af75d93b2c33. Other resources are available at https://brill.com/view/book/9789004466807/BP000007.xml?language=en and https://sustainabilitymath.org/.

C. Social Justice as a context for mathematics learning

Social justice issues can be addressed with increasing sophistication from the early grades through college. There is a small but robust collection of materials that can be used as resources in this arena. For example, *Rethinking Schools* has a large collection of relevant materials, specifically those in *Rethinking Mathematics.* A series of books published by Corwin, (Bartell et al. 2023, Berry et al. 2020, Conway et al.2023), from which we draw a number of our examples, provides lessons at the elementary, middle, and secondary levels.

And, of course, NCTM offers a wide variety of resources, including the *Catalyzing Change* series; see https://www.nctm.org/socialjustice/. Any issues for which there might be disparities in treatment or status of different groups – who gets what kind of health services; what are covid rates, infant mortality rates, etc., for different subgroups of the population? – can be investigated, perhaps as lessons, perhaps as student projects.

See the appendices for additional resources.

Elementary School[4]

Issues and Challenges Particular to Elementary School Mathematics

The contexts and conditions of learning differ at every grade band. As students grow, their memory, capacity for abstraction, social skills, and personal interests can change dramatically. At the same time, the curriculum changes: it becomes more abstract and, in general, more fast-paced. Dealing with all these issues is a major challenge! Here we list some of the challenges particular to elementary school.

Mathematics is and should be experienced as a socially negotiated sense-making activity from children's earliest encounters with it. Young children are naturally inquisitive; they notice patterns, invent ways of talking about quantities, relationships, changes, and space, and develop representations of these. These often go unnoticed by adults. Young children should be encouraged to explore and supported as they discover and reason with and about mathematical ideas. Communicating with others about their understandings and refining them in interaction with others is essential.

In the early grades especially, mathematics learning should be about sense making. Children are discovering who they are. They are putting together their sense of self; they are learning to use language. They are learning their home language, and some may also be learning English as an additional language. In school, they are learning to use multiple modes to communicate their thinking through talk, writing, drawings, graphs, tables, visuals, etc.). They are also starting to learn how to communicate in more formal and mathematical ways. They are coming to see and understand other people, and to see themselves reflecting in how others see and treat them. For all these reasons, classroom discourse must reflect the kinds of considerate, respectful, thoughtful and reflective discourse we hope students will engage in, in all arenas, throughout their lives.

[4] "Elementary mathematics" refers to mathematics in grades K-5, and "upper elementary" refers to grades 3-5.

Three Annotated Elementary School Lessons

Lesson A: Fairness

In the lesson "Tu lucha es mi lucha: Mathematics for Movement Building", students are introduced to decimal concepts and operations as they explore the salaries of the migrant farmworkers who went on strike to raise their pay in the 1960s.[5] Students use choral counting based on the hourly salary of agricultural workers and use the results to solve a real-world problem related to workers' living conditions. This lesson raises issues of fairness and its mathematical documentation in a historical context – but in ways that are directly related to contemporary issues as well as what it means to earn a living wage. The content is aligned to the 5th grade California Mathematics Standards (related to operations on decimals, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction). Also, students use the number line for representing fraction (or decimal) magnitudes and operations. They begin to interpret data distributions to answer questions and pose further questions (Bartell et al., p. 134).

In terms of relevance, the authors of the lesson live and work in California, a state with a very large migrant worker population. Many children in the state have family members who work in agriculture. California celebrates Cesar Chavez Day in the state, a perfect context for the problem. Of course, the issues of fair pay addressed in this lesson are universal. The lesson is developmentally appropriate: Fairness is a concept that children understand at early ages.

Lesson Outline:

Launch:

In a quick write, students are asked to respond to the following question in their journals: *How would you feel if your pay rate was based upon your race, ethnicity, or gender?*
The students do a think, pair, share with a partner, and a whole group share out.
The students look at images of grapes and do a notice and wonder.
The students answer questions about their knowledge of migrant farm work.
The students are shown the video about Larry Itliong, a Filipinx farmworker and organizer of the United Farmworkers Union.[6]
The students are asked to respond verbally to the following question: *What are you wondering about after viewing the video?*

Explore:

Filipinx and Latinx laborers worked in separate fields and lived in separate camps. The Latinx laborers were paid $1.40 per hour but the Filipinx laborers were paid $1.25 per hour.
Students explore the Filipinx hourly wage, with the goal of addressing the question, *Is this fair?*
The students engage in a choral count, starting with $1.25 and adding $1.25 each time. The amount $1.25 is the hourly wage that Filipinx workers were paid. The teacher arranges the

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[5] The complete lesson "Tu lucha es mi lucha: Mathematics for Movement Building" can be found in *Upper Elementary Mathematics Lessons to Explore, Understand, and Respond to Social Injustice* (Bartell et al., 2023).
[6] See https://ca.pbslearningmedia.org/resource/filipino-american-farmworkers/asian-americans-video/.

numbers in an array that has six columns, in order for students to more easily identify the amount paid for 12 hours of work – one day's pay (see Table 2).

Table 2

1.25	2.50	3.75	5.00	6.25	7.50
8.75	10.00	11.25	12.50	13.75	15.00
16.25	17.50	18.75	20.00	21.25	22.50
23.75	25.00	26.25	27.50	28/75	30.00
31.25	32.50	33.75	35.00	36.25	37.50
38.75	40.00	41.25	42.50	43.75	45.00

The students spend time looking for patterns in the table, sharing their observations and wonderings.
After discussing patterns in the table, the students are given the following problem to work on.

> *Filipinx Farmworkers Problem*
>
> In the early 1960s, manongs, Filipinx farmworkers, were paid $1.25 per hour of labor. On average, farmworkers were expected to work 12 hours a day, 6 days a week. How much did the manongs earn in one day? The manongs had little money and relied on low-income housing in which rent was $50 a month.
>
> How much would the manongs make in a month? How much money would the manongs have left over after paying rent? What questions do you have about this situation? Is this a fair wage?

The students work in groups using choral counting table to solve the problem. They record their group's solutions on chart paper.

Summarize:
One representative from each group shares the group's solution to the problem, along with their questions and comments. The class discusses them.

Taking Action:
With what they have learned, students can research the current hourly pay of migrant farmworkers and analyze the relationship between their pay and their living expenses. Students can write letters to the United Farmworkers Union to find out about current conditions for migrant farmworkers.

Students could explore wages and salaries of jobs in their communities and/or jobs they are interested in. Students analyze the relationship between wages and living expenses.

Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:

This lesson involves students engaging in number and operations, and analysis as they make sense of historic farmworker salaries and living expenses. We note that choral counting extends typical ways of counting out loud by charting the count and provides opportunities for students to consider what they notice about the number sequence, leveraging student ideas to build conceptual understanding of number (Franke, Khazemi, and Turrou 2018). Additionally, it is a routine that promotes discourse and justification conducive to mathematical understanding.

Epistemology:
This task supports students in making sense of quantities and how numbers grow, in a real-world context. That context enables students to connect numbers to real-world situations in meaningful ways, and to see that mathematics is relevant and useful for understanding the world around us.

Dispositions:
This task allows students to make sense of real-world data in an historical and contemporary context. They work in groups to share observations and wonderings regarding choral counting of hourly wages. The students use what they learned from the small group and whole group discuss what a fair wage is. All of this contributes to students' sense of mathematical agency, where they see themselves as mathematical sense makers.

Ethics:
By analyzing the wages of migrant farmworkers, students will have a better understanding of the impact of wages on living conditions and the ways in which the people who are responsible for producing our food are treated.

Developmental Needs:
For children in upper elementary school, perceiving linear growth patterns can be a challenge. The ideas of proportionality underlying the ways in which hourly salaries add up are not obvious to many children at this point (e.g., that a 12-hour day's salary is twice the salary earned after six hours, etc.). That is why students need time to explore the patterns in Table 2. And, learning to link their numerical observations to real-world conditions in meaningful ways offers additional challenges. Addressing these challenges prepares students to make meaningful use of school mathematics.

Language:
In exploring issues like the ones in this lesson, students need to be able to use language to communicate in ways that make sense to them. The lesson does not ask students to use particular mathematical language; informal language and others forms of expression are supported and valued. The lesson asks students to share their ideas in words and gestures. It includes pictorial representations to support students' sense making. The teacher's responsibility is to support the students' engagement and help them use both informal and more formal mathematical language in their sense making. Students can actively compare and contrast different mathematical approaches (see Appendix B, MLR7-Compare and Connect).

<u>Lesson B: Climate Justice</u>

In the lesson "Examining Air Quality", students explore what air quality measurements are and how they relate to whether air is safe to breathe or not.[7] The students analyze air quality maps from different locations and decide whether it is safe to go outside.

Lesson Outline:

Launch:
Students are shown a map of California with air quality readings. In a whole group, students discuss what they notice and wonder about the map, including what the numbers and colors indicate. Students discuss the meaning of the air quality ratings and why they are important.

Explore:
Students work in groups. Each group receives a different air quality map to investigate. Students take turns sharing what they notice and wonder about the maps. Students return to the whole group share group exploration findings.

Summarize:
Students take a position and support it with evidence. Students look at an air quality map for their neighborhood and determine whether the air is good, moderate, or unhealthy. Students place stickers on an air quality number line to indicate where they think today's air quality would be located.

Taking Action:
Students can talk about air quality with their parents, caregivers, and the adults in their lives. Students can pay attention to news reports about air quality.

Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:
This lesson involves very young students engaging in number and operations, measurement, and data collection and analysis as they make sense of the information on the air quality maps. As the lesson notes, "Many children in kindergarten are learning numerals, number names, and rote counting by ones. They are making sense of quantities and how our number system is ordered, exploring ideas like how the more digits a whole number has, the bigger it is. Alongside this, they are reasoning about the meaning of amounts and are learning concepts to compare numbers such as 'more than' and 'fewer than.'"

Epistemology:
This lesson supports students in seeing that different ways to represent real-world phenomena (maps, numbers) can help them to understand the world around them. The classroom discussions help students see how exchanging ideas with others helps to understand things better.

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[7] The compete lesson "Examining Air Quality" can be found in *Early Elementary Mathematics Lessons to Explore, Understand, and Respond to Social Injustice* (Koestler et al., 2022). Resources for the lesson can be found at resources.corwin.com/TMSJ-EarlyElementary.

Dispositions:
This task allows students to make sense of real-world data that impacts their lives. They work in groups to share observations and wonderings regarding the air quality maps. They discuss as a whole class similarities and differences between the maps they explored. Lastly, they use what they learned from the small group and whole group discussions to make claims about the air quality of their own region. All of this helps students to see themselves as agentive learners.

Ethics:
By analyzing the data, students will discover disparities in the air quality of regions that are related to socio-economic factors. Students will also be able to understand for themselves what the air quality ratings mean and how to behave based upon those ratings.

Developmental Needs:
From the earliest of days in school, students need to believe that mathematics makes sense, is related to the real world, and is doable. This task allows students work together to make sense of the data in the maps, compare and contrast their findings with other students, and make conjectures related to the quality of air they breathe in their own communities.

Language:
This task introduces students to using a range of mathematical and other terms that require careful use of language. Students need to actively use and ground meanings for mathematical terms in actions, concrete objects, and/or representations. What does "more" mean, for example, and how do different meanings correspond to the written numerals or the colors used to represent air quality on the map? Sorting out such issues requires discussion (see Appendix B, MLR8: Discussion Supports), and attention to using vocabulary with meanings that are appropriate for very young students.

Lesson C: Social Justice

Access to Parks
Who has access to parks and who doesn't? Why do some communities have lots of park space while other nearby communities have almost none? Should there be more parks, and if so, where should they be located? What kind of services should they offer? UCLA's *Data for Democracy in LA* project offers a template for discussions and a set of data representations to consider. These can be used to investigate inequality related to parks or as a template for issues that could be examined in readers' communities.

The intentions of the lesson are (a) to draw on data to support conversations around inequality that are based in data and (b) to critically examine the data sources and trustworthiness of the data. For example, the students can engage with a table that shows the median income and park acreage for 10 local communities. They are asked to consider where the data comes from and whether they should trust it. Students can compare different data representations to see how the representations might highlight different aspects of the data. They can engage with a map that shows where parks are located and shows median income, and then explore a representation of two of those communities that vary in median income and the park acreage for each. The data brief includes a wide range of both quantitative and qualitative data to support students in considering what can be learned from these different kinds of data.

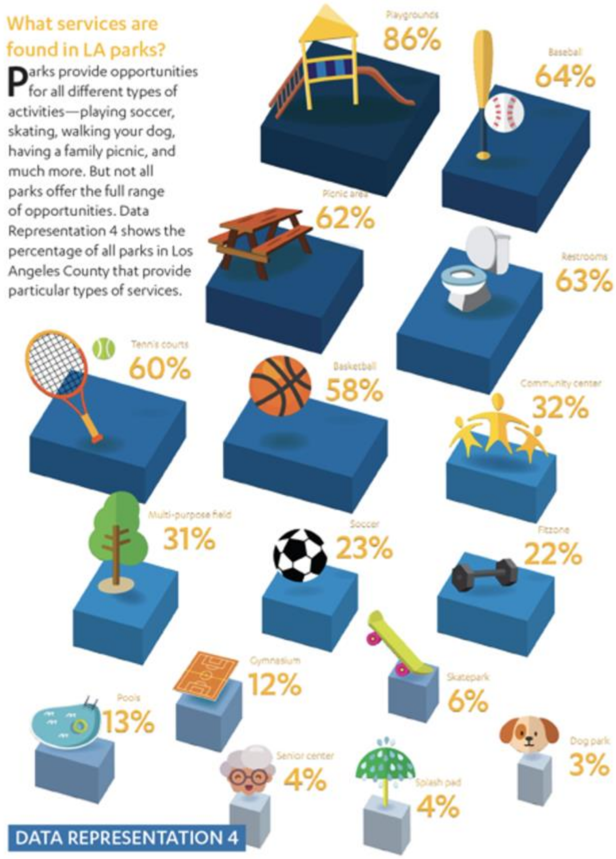
The complete lesson, Parks in Los Angeles, can be found here. The website for the project, Data for Democracy in LA, can be found here.

Lesson Outline:

Launch:
Students are asked about the parks in their neighborhood. How far do they have to walk to get there? What is available at their parks? Does they have soccer fields or bathrooms? Do the students ever drive far to get to a park they really like to play at? What does that park have that they like? The intention here is conversation; there is no need to collect specific data from the students in this opening but rather learn what they are interested in related to parks.

Explore:
The teacher shares the representation showing the services available in parks in Los Angeles (or replaces it with one from their community). Students are asked what they notice and what they wonder about. What surprises them? Why might that be? (See figure 4.)

Figure 4: Services available in L.A. parks



The class pursues comparisons of what services are most available and which least available. They might consider questions like, "if only 63% of parks have bathrooms or 23% have soccer fields where do you think the parks are located that have bathrooms or soccer fields? Are there certain features that you think are essential for all parks?"

Students consider questions such as: which park service is found almost twice as often or if there are 1,587 parks represented in this data set, how many more parks have baseball fields than bathrooms? For a mathematical challenge that involves proportional reasoning, they can consider this issue: if you wanted to build 10 new parks with the same representation of services that now exist in LA parks what would services would the parks have? What if you were building 100 more parks?

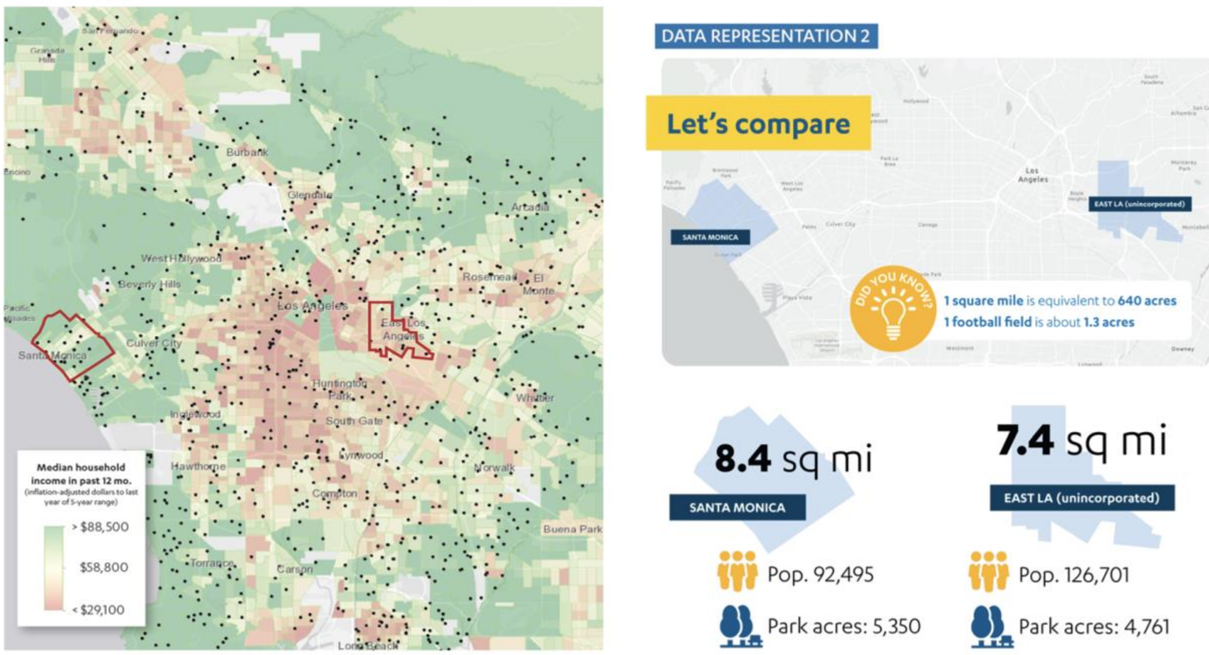
Taking action:
There are opportunities to learn more about the parks in the community. Students can choose a question they want to answer. For instance: Which parks in our community have bathrooms? Where are they? Are they in communities where income is higher? Are they in communities that have more park acres? How can they find out? Where would they look or who could they ask?

Summarize:
Students are asked to make sense of these data in relation to their own parks. What services do they think would be most important for their community? Why? Do they think other students at

their school would agree with them or do they think different students are going to want to do other things at their parks?

Extension:
The map in Figure 5 shows the median income of communities in Los Angeles and where parks are located. Two communities with varying incomes are highlighted and shown more closely in the second representation. Students can discuss what they notice.

Figure 5: Park locations and median incomes



Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:
This lesson allows the teacher to choose their mathematical focus. It asks students to make sense of a representation and extend the data in ways that ask for them generalize. The data conversation can be used to compare percentages, to consider percentages in relation to quantity and to reason proportionally.

Epistemology:
These data ask students to consider how space in communities is designated and who has access to what kind of space, starting with a space that students would most likely have experienced. It asks students to wonder why that may be, what space would be important for everyone and who makes those decisions. It is important to situate parks historically and recognize that parks sit on indigenous lands. The lesson helps students to make connections between school mathematics and its use to understand "real world" issues.

Dispositions:

The goal of these lessons it to provide students the opportunity to see how to use data and mathematics to make sense of their world and take action. The intention is inquiry that is driven by students and conversation that asks them what they think is important and why and what they can do about it.

Developmental Needs and issues:
Elementary students are filled with wonder and are creative problem solvers. These lessons take advantage of that and enable students to draw on their experiences and understandings to make sense of data, have conversations about inequality tied to those experiences, and wonder about different communities. It asks students to consider that not all students experience the world in the same way or agree with each other, and it helps them to refine their thinking in conversation with others.

Language:
In exploring issues like the ones in this lesson, students need to be able to use language here in ways that make sense to them. It does not ask students to use particular mathematical language; informal language and others forms of expression are supported and valued. The lesson asks students to share their ideas in words and gestures (see Appendix B, MLR8: Discussion Supports). It uses a pictorial representation to support their sense making. The teacher's responsibility is to support the students' engagement and help them use both informal and more formal mathematical language in their sense making.

Middle School

Issues and Challenges Particular to Middle School Mathematics

Many middle school mathematics students begin to lose interest in the discipline and will, when possible, make the choice to leave mathematics as soon as they can. The challenge is to maintain student interest. Addressing relevant "real world" issues in meaningful ways is one way to do so, especially if the students can draw upon their real world knowledge to help make sense of the issues they are studying.

There are many considerations from the student's point of view. Students are their "whole selves" when they enter the classroom – they don't leave their moral selves, their social selves, or their other personal commitments at the door. In developmental terms, middle school students are developing and refining their sense of moral reasoning. Peer opinions matter a great deal to them; there are issues of attitude and confidence, especially given gender and racial stereotypes. Hormones begin to kick in, along with myriad insecurities and issues of identity formation. As a handout from the National Association of School Psychologists (2018) notes, middle school students are simultaneously coping with physical, social, and emotional changes (think hormones!). Their identities are in flux and are often fragile.

The school context also changes significantly. In elementary school, most students stay with one or two homeroom cohorts for much of the day. In middle school, they often spend time in disciplinary silos – math class, science class, English, social studies, etc. There are also age-specific issues of language. By the time students enter middle school they have acquired their first language (Valdes, 2001). This can serve as a resource; whether or not that first language is English, everyday language is a resource, in general, for problem solving and sense making and, specifically for exploring new topics, sharing beginning explanations with peers, and sorting out misunderstandings. Learning to communicate about new mathematics in middle school (for example sharing one's reasoning in topics such as algebra), constructing, revising and critiquing new kinds of arguments, and learning to use more formal registers are some of the new goals for math instruction in middle school, and those multiple goals need to be negotiated carefully.

While elementary school teacher preparation has a significant focus on the "whole child" and developmental issues, and secondary school preparation has a strong disciplinary focus, effective teaching in middle school – actually, all grades – calls for significant attention to both. Often, middle school mathematics teachers have little preparation for, or professional development in, addressing potentially controversial social issues, both from the curricular side (these are complex, value-laden issues of modeling) or the interpersonal side (where managing the emotions that can emerge in fraught discussions can be a significant challenge). Curricular pressures begin to accelerate, with more and more content coverage each grade, and high stakes testing. These are challenges to be faced if room is to be made for including the kinds of tasks that support civic discourse that are discussed in this report.

Three Annotated Middle School Lessons

Lesson A: Fairness

In the lesson "Gender gap and pay", students discuss differences between the salaries of those who identify as males and females by examining the percent change of salaries across various careers.[8] They then do a more in-depth study of a particular career of their choice, in order to investigate the variables potentially responsible for the variations. (These variables can include experience, postsecondary education, age, etc.) Students use a range of mathematical representations to explore gender pay gaps and discuss issues related to them.

Lesson Outline:

Launch:
Students work in small groups to determine the median salaries of males and females in a range of professions. They explore the use of median as opposed to other measures (e.g., mean) to fairly represent a body of data, and learn to compute salary differences in terms of percentages. They discuss possible causes of gender pay gaps.

Explore:
Students select one job category for intensive exploration. They then examine scatterplots of male and female salaries in their chosen profession over time, and determine best fit lines to represent salary trends. Students make posters to demonstrate what they have found.

Summarize:
The class engages in an organized discussion of the posters, after which it considers the efficacy of the mathematical representations and discusses various parameters used in those representations.

Taking Action:
Students are encouraged to discuss possible career choices and the impact of their explorations on those choices. Extension activities include researching further job categories, reflecting on the career of Ruth Bader Ginsburg, and doing research on equal pay legislation.

Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:
The mathematical goals for this lesson are for students to be able to use data from their own research to create a scatter plot and two-way table. Students also look at the specifics of their scatter plots and two-way tables, such as clustering data, outliers, positive or negative association, and lines of best fit. Along with creating the scatter plots and two-way tables, students interpret their meaning in relation to the social justice topic, which helps them learn

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[8] The complete lesson "Gender gap and pay" can be found in *Middle School Mathematics Lessons to Explore, Understand, and Respond to Social Injustice* (Conway et al., 2022). Resources for the lesson can be found at resources.corwin.com/TMSJ-MiddleSchool.

about the association between the variables. The students also consolidate understandings of fractions and percent differences and study the properties of mathematical representations.

Epistemology:
Classroom discussions open up explorations of the factors that influence how salaries are set, and social factors that may determine salaries by gender. This leads to the possibility of developing understandings regarding the social construction of inequities.

Dispositions:
The activity provides students with tools for inquiring into possible careers of interest, and for understanding social issues related to them. This has the potential to foster an increased sense of agency.

Ethics:
Wage gaps and their causes are significant ethical issues. This lesson provides a solid data-based way to enter into discussions and problematize the current state of affairs.

Developmental Needs and Issues:
Middle school students are first learning to grapple with multivariate phenomena and the idea that there may be a range of causes for single-variable outcomes. Coming to grips with the fact that phenomena are not simply deterministic, but that there are meaningful trends, is a significant challenge.

Language:
Before embarking on finding solutions for this lesson, students first need to have the time and the opportunity understand the problem well enough to dig into it (see, e.g., Appendix B, MLR6-Three Reads). This task provides opportunities for to compare and contrast different mathematical approaches (see Appendix B, MLR7-Compare and Connect). Instruction can also support students actively revising their ideas, explanations, or arguments (see Appendix B, MLR1-Stronger and Clearer).

<u>Lesson B: Climate justice</u>

The lesson "The Mathematics of toxic air emissions" explores the impacts of toxic air emissions on an economically disadvantaged neighborhood in the Midwestern United States.[9] In this lesson, students are expected to explore different representations of numerical data and what information can be easily gleaned from those representations. Specifically, students are expected to use the information from a table containing toxic air emissions and average income by zip code, choose an appropriate graph to represent the data, graph the data, and finally, make observations and conclusions based on the data.

Lesson Outline:

Launch:
Students read a worksheet with data concerning toxic air emissions in Cincinnati, Ohio. They discuss issues raised by the information in the worksheet and then consider potential uses of mathematics for representing or misrepresenting important information.

Explore:
The students discuss possible relationships between categories of data in the table (zip codes, toxic air emissions, and average household income). Students create a graph using "average household income" and "toxic air emissions," and then explore particulars: what is the average income in the area with the largest amount of toxic air emissions, and with the smallest amount? They learn about the impact of the emissions on people's lives, and discuss the correlations in the data.

Summarize:
In the summary discussion, students review what they learned from their graphs, and how they discovered the inequitable patterns. They are led to explore data for their own neighborhoods and discuss the implications.

Taking Action:
Students are encouraged to write letters to their local and federal environmental protection agencies about areas that have bad air quality, and to raise awareness of the issue to others in the school and in the community through local and social campaigns about clean air.

Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:
The mathematics in this lesson involves students translating from one visual representation of numerical data to an alternative representation of those data. Specifically, students are expected to use the information from a table containing toxic air emissions and average income by zip code, choose an appropriate graph to represent the data, graph the data, and finally, make

[9] The complete lesson "The Mathematics of toxic air emissions" can be found in *Middle school mathematics lessons to explore, understand, and respond to social injustice* (Conway et al. 2022). Resources for the lesson can be found at https://companion.corwin.com/courses/TMSJ-middleschool.

observations and conclusions based on the data. More broadly, this kind of information (that the environmental quality of life may well be a function of income) may well be new to students.

Epistemology:
These data complexify simple assumptions about life. People have places to live. We know some places are "nicer" than others in some ways. But to realize that where you live can have a direct impact on your health because of environmental factors may be new, and may open up different ways of thinking about social justice.

Dispositions:
The activities described in "taking actions" may help students to see that it is possible to act on information you learn, giving them more of a sense of agency. The lesson as a whole may reinforce the idea of mathematics as a discipline that helps to make sense of, and act on, aspects of one's personal life.

Ethics:
The information that basic health risks may correlate with income may be a revelation, information that needs to be known and acted upon. The two articles mentioned above[10] indicate that this is an arena ripe for exploration.

Developmental Needs and Issues:
Middle school students are moving from simple right/wrong views of complex issues to beginning understandings of their complexities and the tensions involved when trying to resolve such issues. They need to learn and believe that their understandings and actions can have societal impact, and also to find a way to put such understandings in perspective.

Language.
In exploring issues like the ones in this lesson, students need to begin by using language to communicate in ways that make sense to them; informal language and others forms of expression need to be supported and valued. The teacher's responsibility is to first support the students' engagement and help them use both informal and more formal mathematical language in their sense making (see Appendix B, MLR2, Collect and Display). Beyond those initial sense making discussions, instruction can support students in learning to revise their arguments (see Appendix B, MLR3-Critique, Correct, and Clarify).

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[10] Articles can be accessed here: https://www.nytimes.com/2022/03/09/climate/redlining-racism-air-pollution.html and https://www.washingtonpost.com/climate-environment/2022/03/09/redlining-pollution-environmental-justice/.

<u>Lesson C: Social Justice</u>

In the lesson "Driving while Black and Brown," students do a probability simulation to determine whether or not data from police traffic stops represents a case of racial profiling.[11]

Lesson Outline:

Launch:
Students are asked if they have any experience with being racially profiled or if they know anyone who has been racially profiled. Students share their experiences and/or knowledge of racial profiling

Explore, Part 1: Finding California's Racial Breakdown
Each group of students is given a small bag with colored cubes to represent the racial breakdown of California. The bags contain nineteen tan cubes (for whites 38%), nineteen red cubes (for Latinx 38%), seven blue cubes (for Asians 14%), 3 black cubes (for African-Americans 6%), and two green cubes (for others 4%) to approximate California's racial proportions. Do not tell the students the total number of cubes or how many of each color there are.

Students pick one cube from the bag without looking. They record its color on the chart (tally marks work well), and replace the cube. Repeat this 100 times. After every 10 picks, the students record totals on a chart. Each line of the chart is the cumulative total of picks. Students are told that they are conducting an experiment (picking/replacing 100 times), collecting data (recording each pick), and analyzing data (determining for their simulation how many cubes of each color are in the bag, the total number of cubes, and the California racial/ethnic percentages).

It is important for the students to record the fraction and the percentage of each race/ethnicity for every 10 picks on the chart.

Each group of students is asked to provide thorough explanations for the following questions.

1. What do you think is in the bag? Why do you think this?
2. What happened as you picked more times, and what do you think would happen if you picked 1,000 times?

Explore, Part 2: Investigating Driving While Black and Brown
Students are provided the following data from the Los Angeles *Times* analyzing traffic stop data from 2012-2017: Of 378,000 drivers stopped by the California Highway Patrol (CHP), 151,200 were Latinx.

During the same period of time the Domestic Highway Enforcement Team, (DHET), a team of Los Angeles County sheriff's deputies who cruise the 5 Freeway, stopping motorists in search of cars carrying drugs, pulled over about 13,333 people. Of those stops, 9,200 were Latino.

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[11] The complete lesson "Driving while Black and Brown" can be found in the book *Rethinking Mathematics: Social Justice by the Numbers* (Gutstein & Peterson 2015). Resources for the lesson can be found at https://drive.google.com/drive/folders/11Bb-ZzJZTcWe9ddbnP84bcguTBi74EKa.

Students are asked to use what they learned in the first experiment to set up their own simulation of this situation using cubes. They are instructed to pick and replace a cube 100 times, record the data, and calculate the results of simulating 100 stops.

Students are asked to address these questions in groups:

1. What percentage of the motorists pulled over by the CHP were Latinex?
2. What percentage of the motorists pulled over by the DHET were Latinex?
3. How did you set up the simulation for the investigation (how many "Latinex" cubes and how many total) Why did you choose those numbers?
4. In your simulation, how many Latinx were picked out of 100 picks, and what percentage is that?
5. Do the results of the simulation investigation support the claim of racial profiling? Why or why not?

   Combine individual groups' results and analyze as a whole class.

Summarize:
Students are asked:

1. What did you learn from this activity?
2. How did the mathematics help you do this?
3. Do you think racial profiling is a problem, and if so, what do you think should be done about it?
4. What questions does this project raise in your mind?

The lesson ends with whole-class discussion of these issues.

Taking Action:
Students use responses from summary question 3 to develop a plan of action around racial profiling. Students can reach out to community organizations that deal with police accountability. Students can contact their local law enforcement organization to determine their policies with respect to racial profiling. Students can contact their elected officials to advocate for laws against racial profiling.

Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:
This content focus of this lesson is probability. Students engage in a probability simulation in which they randomly draw colored cubes from a bag, with replacement, to determine the contents of the bag. This indicates how the distribution of results from random draws corresponds to the distribution of the whole population. Issues of experimental versus theoretical probability are addressed through this lesson. These understandings are used to assess the likelihood that police stops are random.

Epistemology:
In the lesson, the students are given a scenario related to police stops by race, and are required to create their own probability simulation to determine of the scenario represents a case of racial profiling. This helps students to understand that it is possible to use mathematics to assess the

likelihood of certain “real world” events, and to uncover biases in an objective and scientific way. Even the notion of “likelihood” is challenging.

Dispositions:
While many students may have opinions about racial profiling based upon their personal experiences or what they have seen or read in the media, it is important that students base their decision about whether or not the scenario represents a case of racial profiling on the mathematics. This kind of unit can help develop a sense of agency, where students see that they can uncover injustice in ways that are compelling.

Ethics:
It is possible that students may come to different conclusions based upon the results of the probability simulation. It is important to have students base their conclusions on the mathematics involved. At issue here are underlying ethical commitments to the unbiased application of the law and the neutral application of mathematical methods to determine what can mathematically be judged to be biased or unbiased.

Developmental Needs and Issues:
The issues dealt with in this unit are challenging personally and mathematically. On the personal side, dealing with issues of bias is extremely difficult in terms of identity, especially during times of identity formation. Seeking somewhat objective tools to determine bias is not a “natural” thing to do. Then, the entire framing of probabilistic issues, which do not “prove” claims but instead discuss their likelihood, is difficult to understand.

Language:
In exploring issues like the ones in this lesson, students need to begin by using language to communicate in ways that make sense to them; informal language and others forms of expression need to be supported and valued. The teacher’s responsibility is to first support the students’ engagement and help them use both informal and more formal mathematical language in their sense making (see Appendix B for another way to support students active sense making and problem while attending to language, MLR5-Co-Craft Questions and Problems). This can be done orally or in writing, to support revising. Beyond those initial sense making discussions, instruction can support students in learning to revise their arguments (see Appendix B, MLR3-Critique, Correct, and Clarify).

Secondary School

Issues and Challenges Particular to Secondary School Mathematics

The mathematics students encounter in secondary school becomes increasingly abstract and technically difficult. This presents an opportunity and a challenge. The opportunity is that secondary school mathematics is remarkably powerful: there are tremendous opportunities for sense-making and meaningful applications to important real-world problems. The challenge is that students do not often experience mathematics this way: many students lose interest in the discipline and make the choice to leave mathematics as soon as they can. Addressing relevant "real world" issues in meaningful ways is one way to maintain student interest.

There are many considerations from the student's point of view. Students are their "whole selves" when they enter the classroom – they don't leave their moral selves, their social selves, or their other personal commitments at the door. In developmental terms, students are developing and refining their sense of moral reasoning. Peer opinions matter a great deal to them; there are issues of attitude and confidence, especially given gender and racial stereotypes. And, of course, adolescents rebel. The realities of the real world (job possibilities, racism and issues of social justice, for example) really begin to hit home for students. On the one hand, this means that the careful consideration of such issues opens up possibilities for meaningful discussions in mathematics classrooms. On the other hand, such issues are delicate and must be addressed with appropriate care.

There are age-specific issues of language. By the time students enter secondary school they have acquired their first language. This can serve as a resource, whether or not that first language is English. At the same time, communication in mathematics classrooms has its own demands, and those need to be negotiated carefully.

The secondary context differs substantially from middle school. Tracking in mathematics classes becomes a more significant issue at the secondary level, with implications for equity and for student identities. Most secondary mathematics teachers teach mathematics only. Typically they have little preparation for, or professional development in, addressing potentially controversial social issues, both from the curricular side (these are complex, value-laden issues of modeling) or the interpersonal side (where managing the emotions that can emerge in fraught discussions can be a significant challenge). In addition, there are greater curricular pressures – an increasingly fast-paced curriculum, demanding more and more rapid content coverage, and high stakes testing that tends to narrow the focus of instruction to what is tested. All of these are challenges to be faced if room is to be made for including the kinds of tasks that support civic discourse that are discussed in this report.

Three Annotated Secondary School Lessons

<u>Lesson A: Fairness</u>

In the lesson "Do Postal Codes Predict Test Scores?", students explore the edgap.org website, which provides graphical representations of average high-school SAT and ACT scores, median household income, and other socioeconomic indicators in a range of districts across the U.S.[12] The students then choose variables of interest to them and explore correlations between those variables and SAT and ACT scores.

Lesson Outline:

Launch:
Students work in groups with an internet connected device to explore the edgap.org website. They take 10 minutes to write down observations and wonderings. Groups share their observations and wonderings. To assure basic understandings, students are probed as to what the different numbers and colors on the map represent.

Explore:
Students work in groups to decide what data they want to analyze, from the four areas provided by edgap.org: median household income, percent unemployment rate, percent of children in 2-parent household, percent of college educated parents.

The students then use the edgap.org website to create a table of 20-30 schools of interest, recording the ACT/SAT score and their variable of choice. They use the data from the table and technology to create a scatterplot and model for their data, after which they determine the correlation coefficient r. The data points on their scatterplots are labeled by the school name.

Summarize:
Students do a gallery walk to share findings. Students notice similarities and differences in their graphs, models, and correlation coefficients.
Students discuss any economic and/or racial trends and disparities in the data.

Taking Action:
Students brainstorm who they might write a letter to in response to their data analysis. They work together in groups to compose letters to the people on the brainstorm list.

Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:
This lesson taps into a range of knowledge related to scatterplots as a mechanism for analyzing the association between two variables. The visual representations of different correlations allow students to see how strong the correlations are (the visual counterpart to correlation coefficients).

Epistemology:

[12] The complete lesson "Do Postal Codes Predict Test Scores?" can be found in *High School Mathematics Lessons to Explore, Understand, and Respond to Social Injustice* (Berry et al. 2020). Resources for the lesson can be found at https://resources.corwin.com/tmsj-highschool/student-resources/chapter-7/lesson-74-do-postal-codes-predict-test-scores.

This task allows learners to become aware of biases in standardized testing that they may not have been aware of. This is an important understanding – that numbers like test scores are not absolute measures of someone's knowledge or potential, but that a number of factors outside a person's control may affect someone's test scores. Because the overall task is multivariate, it illustrates the complexity of working with real-world issues, rather than the standard "there is one right answer to every math problem" impression that students typically develop from the curriculum. Students get to see that one variable can have *some* impact on another, but only in the aggregate, not in a uniform way.

Dispositions:
This task reinforces a number of dispositions essential for the thoughtful modeling of real-world phenomena. The students get to weigh competing evidence. Which test (if either) reflects proficiency? What kind of proficiency? Which variables matter, in which ways? Students get to play with variation as a way of approaching complex phenomena. The learners create a data table of 20-30 data points using their variables of choice, graph their data, and use technology to determine the correlation coefficient r. Different choices of data points result in different values, and students will need to sort out these differences. Similarly, they will need to sort out the impact of the four different variables they compare collectively – median household income, % unemployment rate, % of children in 2-parent households, and % college educated parents. The learners do a gallery walk to share findings and look for patterns in class graphs. In doing so they need to listen to each other and sort out commonalities and differences.

Ethics:
The data are likely to challenge some fundamental assumptions about testing. Standardized tests are supposed to be objective measures of student ability, but the data seem to suggest that students from higher income backgrounds score higher on standardized tests than students from lower income backgrounds. That finding can raise all sorts of ethical issues. If the tests are unfair, should they be thrown out? Is there a way of compensating for test bias? If the compensation is statistical, will it be fair to individuals? What are more equitable ways to determine what students know?

Developmental Needs and Issues:
Students learn to collect and organize data that is important to themselves. They grapple with complexity and make sense of data. In doing so, they may have increased opportunities to develop self-efficacy. These data may well be relevant to the students personally (or their friends), if they see themselves reflected in the data pool. The lesson may raise questions for them about their own biases and perceptions about testing, and about society's biases. And, it prepares them to confront subtle issues of bias in society, and to discuss such issues thoughtfully.

Language:
Before embarking on finding solutions for this lesson, students first need to have time and the opportunity understand the problem well enough to dig into it (see, e.g., Appendix B, MLR6- Three Reads). This task provides opportunities for to compare and contrast different mathematical approaches (see Appendix B, MLR7-Compare and Connect). The teacher can ask students to generate mathematical questions or problems for the situation, using this lesson to create space for students to produce the language of math questions themselves and develop meta-awareness of the language used in mathematical questions and problems (see Appendix B, MLR5- Co-Craft Questions and Problems).

<u>Lesson B: Climate justice</u>

In the lesson students examine 100 years of real data from Alaska that chronicle when seasonal ice cracked enough to allow for bridge or other construction.[13] They use scatterplots and various algebraic mechanisms to model the timing of ice breaks and make predictions.

Lesson Outline:

Launch:
Students are introduced to the Nenana Ice Classic, including a description of the historical origins of the wagers on the river's ice block breaking up, and they are given a brochure with more details. They then visit a website that gives information related to this year's contest. The students discuss and explore patterns in the data. They are then charged with predicting the day and time when the ice in the Tenana River at Nenana will break. They work in groups to make predictions.

Explore:
Students use the data provided from the brochure and website to create a graphical display and model that can be used to predict when the ice will break the following year. They use technology to plot their data and fit an appropriate model to their scatterplot. Students discuss trends that they see in the data and their models. They make conjectures regarding to why the ice seems to be breaking earlier as time progresses.

Summarize:
Student groups share their scatterplots and models. They interpret the parameters of their models in the context of climate change in Alaska.

Taking Action:
Students brainstorm actions they can take to deal with climate change (e.g., recycling, using reusable products, using public transportation). Students choose which action they want to take and complete that action as a final project.

Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:
This lesson involves the use of equations and inequalities to make predictions. Students analyze the relationship between two variables using statistical procedures, including determining the quality of a linear fit to data, generating least-squares regression lines, finding the correlation coefficient, an exploring the difference between correlation and causation.

Epistemology:
The task helps students understand that prediction is not an exact science, but that models enable one to get a grasp of complex phenomena. That is the essence of applied mathematics and

[13] The complete lesson "Climate Change in Alaska" can be found in in *High School Mathematics Lessons to Explore, Understand, and Respond to Social Injustice* (Berry et al. 2020). Resources for the lesson can be found at https://resources.corwin.com/tmsj-highschool/student-resources/chapter-6/lesson-62-climate-change-in-alaska.

modeling. Students learn to sort through complex phenomena to find variables that seem to matter, and to test whether they actually do matter (and to what degree) in empirical fashion. This is a form of dealing rigorously – but not exactly, as in classical mathematics – with issues of modeling and prediction.

Dispositions:
This task reinforces a number of dispositions essential for the thoughtful modeling of real-world phenomena and for doing sense-making with data. Students come to deal with multivariate phenomena, which is often a challenge. They work in groups to make predictions, a process that involves collective reasoning and a form of give-and-take that is useful for civic discourse. Specifically, groups share rough draft thinking after thinking about the problem for a few minutes, but before they have come up with a prediction. After each group has shared, the groups continue to work on their predictions using what they heard during the group sharing. Thus reinforces the idea that collaboration is an effective mechanism for advancing understanding.

Ethics:
By analyzing the data, students conclude that the river ice is breaking earlier each year. Students begin to explore the reasons why that is happening. The students explore what the implications for the state are if the ice continues to break earlier each year. This raises a series of issues regarding the impact of human behavior on the climate, and on the particulars of this locale in Alaska. The short HBO video "Climate Change is making this Alaskan Town Fall into the Ocean" underscores the ethical issues involved. Students then explore ways such issues might be addressed.

Developmental Needs and Issues:
The unit helps students develop a sense of communal responsibility for actions, the understanding that individual actions contribute in a cumulative way both to causing problems and potentially to addressing them. They have opportunities to develop self-efficacy both in choosing and analyzing data, and in contemplating the impact of actions taken to address the issues of climate change. They use their knowledge to propose mechanisms for making progress, a contributor to personal identity and empowerment, as opposed to simply arriving at "right answers" on math problems.

Language:
Before embarking on finding solutions for this lesson, students first need to have time and the opportunity understand the problem well enough to dig into it (see, e.g., Appendix B, MLR6-Three Reads). Students can share their oral contributions while the teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. The teacher can provide feedback for students to increase accessibility and also support meta-awareness of language (see, e.g., Appendix B, MLR2-Collect & Display).

Lesson C: Social Justice

In the lesson "How do poor people bank?", students explore how certain kinds of profit-making financial institutions are prevalent in low-income communities, and the mechanisms such institutions use to take advantage of people's difficult financial situations.[14] Students investigate the impact of compound interest on loans.

Lesson Outline:

*Launch***:**
Students share their prior knowledge about interest. Students give examples of when people pay interest and when interest is paid to them. The term interest is defined and an example is given. The concepts of compound interest and annual percentage rate are introduced. Students are given the compound interest formula $A= P(1 + r/n)^{nt}$. They are given an example that requires them to use the compound interest formula.

Next, students are introduced to ways that compound interest can work against them. Examples include Check Cashing establishments, Pay Day Loans, Rollover Loans, Rapid Tax Refunds, Money Transfer Agents, and Credit Cards.

Explore:
Students are given a set of 6 problem scenarios to solve involving compound interest to solve in groups.

Summarize:
Students share solutions to the problems they solved. They respond to the following questions: What is the problem with this system? How is this bad for people? How does this keep people poor?

Taking Action:
Students write letters to elected officials arguing for the need for regular banking institutions in all communities.

Connecting a Mathematics Lesson to Civic Reasoning and Discourse

Knowledge:
This lesson involves Percent, Interest and Compound Interest, Exponents and Exponential Growth. This is both core mathematical content and essential mathematics for financial literacy.

Epistemology:
This task helps students identify the social injustices of banking and lending in low-income communities of color. Working through the issues involved documents the fact that, while the use of mathematics is "neutral" in that it "simply" computes interest, issues of what interest rates are available to which parts of the population are not neutral, and contribute to injustice.

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[14] The complete lesson "How do poor people bank?" can be found at https://www.radicalmath.org/news-1/banking-options-in-low-income-communities.

Dispositions:
The unit indicates that certain questions are fair, if not essential, to ask: How do poor people bank? Why do poor people rely so heavily on expensive financial systems? What are the financial costs of using these methods to bank?

It supports weighing competing evidence: Through the concept of compound interest, students explore the interest rates at a variety of financial institutions, specifically banks vs check cashing/payday loan businesses.

Students examine multiple points of view: They share their results from problem solving and their conclusions about the access to banking services in different communities.
Students listen to others: They share their experiences with banks vs check cashing/payday loan businesses in their communities.

All of this contributes to the understanding that what is learned in math class can be used to understand injustices as they play out in society.

Ethics:
Students confront issues such as the fact that many banks refuse to put branches in low-income communities, causing these communities to rely on check cashing businesses and payday loan establishments, which charge much larger interest rates than banks. This causes disparities in wealth in different communities.

Developmental Needs and Issues:
The lesson contributes to identity development, in that students learn about the financial conditions in their communities. It also contributes to self-efficacy: students are able to analyze the interest rates charged by financial institutions in their communities, a useful if not necessary life skill. More broadly, as a matter of public citizenship, students gain the knowledge to determine whether or not the financial institutions in their communities are serving the needs of the community.

Language:
Before embarking on finding solutions for this lesson, students first need to have time and the opportunity understand the problem well enough to dig into it. Students can share their oral contributions while the teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. The teacher can provide feedback for students to increase accessibility. The teacher can also ask students to generate mathematical questions or problems for the situation, using this lesson to create space for students to produce the language of math questions themselves and develop meta-awareness of the language used in mathematical questions and problems. See Appendix B for the full set of Math Language Routines, a number of which (e.g., MLR6-Three Reads; MLR2-Collect & Display; MLR5- Co-Craft Questions and Problems) could be profitably used in this lesson.

Steps toward implementation, Part 1: Context and Professional development

The large-scale implementation of the ideas in this report will require two types of systemic support. The first relates to the contexts within which teachers work. First, teachers will require "permission" to engage with the kinds of materials discussed here. In some political jurisdictions, the topics emphasized in our examples might simply be off limits. Even if the political climate is supportive, teaching these lessons takes time, a scarce resource. While there is important mathematics in the lessons, and that is typically assessed, connections to "real world" issues are typically not assessed. In consequence, many of the understandings that students develop, no matter how valuable, will not be credited in official reports of student learning. Second, teachers will require supportive materials and professional development. Also, creating a safe space for discussions of potentially contentious issues and helping students explain their thinking and listen to others respectfully as they do so calls for skills and understandings that go beyond those found in the typical mathematics classroom.

In the introductory sections of this report, we identified some of the challenges that the field as a whole needs to address in order for teachers to be willing to move in the directions highlighted in this report. Specifically, NCTM and other groups, including funding agencies, will need to address questions such as:

- What larger institutional support is there for moving in these directions?
- How can such activities be embedded in curricula?
- What kinds of materials support use, with minimum support?
- How can the range of epistemological and dispositional issues highlighted in this report be assessed?
- How can teachers justify the use of these on their own?

Beyond that, there is the question of providing materials and professional development (PD) to support teachers. Minimal PD consists of working through some examples and showing how the patterns of engagement used in these materials (consistent with the principles and practices enunciated earlier in this report) can be applied to all math lessons. But the real bang for the buck occurs when these lessons are used – that's the issue of context, above.

Mechanisms need to be developed that point to ways that teachers, professional learning communities, districts and coaches can take ownership of these ideas and develop them, so that they're not imposed from "above."

For example, this is the type of work that could benefit from lesson study as teachers experiment with implementation of these civic engagement tasks. Lesson study is school-based, teacher driven, collaborative, and student-focused.

Steps toward implementation, Part 2: Funding

The kinds of activities we hope to see take place related to civic discourse in mathematics classrooms are unlikely to occur at scale without some form of institutional backing.

There are various possibilities:

- NCTM might publicize and/or support professional development workshops using the examples in these reports.
- The NSF or other funding agencies might be amenable to similar workshops, especially if endorsed by NCTM.
- It might be possible (with funding) to build video-supported packages that would allow professional learning communities (including school districts) to take on such professional development themselves.
- It might be possible to conduct R&D on materials intended to support these aims. We note that specific materials related to the TRU Framework are available on the TRU web site, https://truframework.org/, and that two new books with tools for professional development (Schoenfeld et al., 2023a,b) are now available.

The "Formative Assessment Lessons" (FALs) built by the Mathematics Assessment Project offers an example of what the right kind of materials can achieve. There were 100 such 2-3 day lessons, available free at https://www.map.mathshell.org/lessons.php. The lessons contain 20-page lesson plans/scripts that help teachers understand the central issues related to student thinking. The materials made a significant difference. Teachers who were provided some PD and then taught 4-6 lessons (10-12 days of instruction) saw content gains of 4.6 months on average (Herman et al., 2014). But, those lessons could not have been developed without significant grant support.

The question is how to find support for projects in these directions, and to point to them. Might NCTM and/or NAEd call for conferences, to generate ideas for materials/PD development? Might various foundations be induced to support pilot projects? Highlighting the issues will call for a communications strategy.

Perhaps California can serve as a model for such a dissemination mechanism. California has the statewide NCTM affiliate California Mathematics Council (CMC) that might be approached to sponsor sessions devoted to civic education and mathematics at their annual conferences. The mathematics specialists at the California county offices of education can provide professional learning opportunities around civic education and math. Lastly, California Mathematics Project can offer professional learning opportunities around civic education and math in their various PD institutes and workshops.

Appendix A: The ICUCARE Framework

From Seda & Brown (2021), with permission.

Appendix B: Resources for Supporting Language (including the "Math Language Routines")

This appendix provides some resources for supporting language in math classrooms. Recommendations from research for attending to language (Zwiers et al, 2017):

- Support sense-making
- optimize student output (both oral and written)
- cultivate conversation, and
- maximize linguistic and cognitive meta-awareness

Important pitfalls to avoid (Moschkovich, 2014, 2016, 2018):

- don't simplify language, AMPLIFY it (Walqui, 2006; Walqui & Bunch, 2019);
- don't focus only on words or vocabulary, instead focus on MATH PRACTICES[15]
- involve and support multiple language functions[16];
- don't focus only on formal language, instead build on students' INFORMAL ways of communicating and carefully consider when and how to support more formal language

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RESOURCE #1: *Math Language Routines*

The eight "Mathematics Language Routines" (MLRs) have been shown to enhance student participation and understanding in mathematics classes. A link to extended descriptions can be found at
https://ilclassroom.com/wikis/178268-ell-mathematical-language-routines.

These routines reflect the collective and collected wisdom of many years of research and practice. They can be enacted in multiple ways. For detailed examples see Zwiers et al (2017); for short summaries, see MELD project materials[17]; for examples specific to middle school math topics, see Illustrative Mathematics (2022) lessons.

When choosing MLRs it is important to:

- Ensure that students are active analytically (reflecting, problem solving, making sense
- Provide opportunities for students to use language in both receptive functions (comprehending, listening) but also in productive functions (talking, writing)
- Provide students with multiple opportunities for revising what they have produced

Below are brief descriptions of those routines.

MLR1 - Stronger and Clearer Each Time-Have students think and/or write individually about a response,

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[15] For an example of a teacher scaffolding math practices, see Moschkovich, 2015.

[16] Language functions, receptive or productive, are what students do with language to accomplish a math task. For descriptions of the language functions associated with the CCSS math practices, see the ELPD framework math tables on pages 20-25 here http://www.ccsso.org/Documents/2012/ELPD%20Framework%20Booklet-Final%20for%20web.pdf

[17] Summary of MLRs and examples from MELD project presentation

use a structured pairing strategy to have multiple opportunities to refine and clarify the response through conversation, and then finally revise their original written response. Throughout this process, the teacher presses students for details and encourages students to press each other for details. Use this language routine to start from and build on informal ways of expressing structure and regularity.

MLR2 - Collect and Display-The teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. This routine provides feedback for students in a way that increases accessibility while simultaneously supporting meta-awareness of language.

MLR3 - Critique, Correct, and Clarify-Give students a piece of mathematical writing that is not their own to analyze, reflect on, and develop. The intent is to prompt student reflection with an incorrect, incomplete, or ambiguous written argument or explanation, and for students to improve upon the written work by correcting errors and clarifying meaning. This routine fortifies output and engages students in meta-awareness.

MLR4 - Information Gap-Give partners or team members different pieces of necessary information that must be used for solving a problem. The intent is to cultivate conversation.

MLR5 - Co-Craft Questions and Problems-Ask students to generate mathematical questions or problems for a situation. The intent is to create space for students to produce the language of math questions themselves and develop meta-awareness of the language used in mathematical questions and problems.

MLR6 - Three Reads-This strategy includes reading a math scenario three times with a different goal each time. The first read is to understand the context. The second read is to understand the mathematics. The third read is to elicit inquiry questions based on the scenario. The intent is to support students in learning to do a close read of a complex math word problems or tasks. For one example see https://www.sfusdmath.org/3-read-protocol.html.

MLR7 - Compare and Connect-Teacher asks students to compare and contrast different mathematical approaches, representations, methods, examples. The intent is to foster students' meta-cognitive and meta-linguistic awareness, and also support mathematical conversation.

MLR8 - Discussion Supports-Any set of talk structures, routines, and moves for supporting participation and engagement. The intent is to support rich and inclusive discussions about mathematical ideas, representations, contexts, and strategies (Chapin, O'Connor, & Anderson, 2009). These can be combined and used together with any of the other 7 routines. The moves invite and incentivize student participation and conversation.

Some MLRs can be used in each of the lessons described in this report:
a) Any lesson that involves reading text can include MLR6-Three Reads. This routine is useful anytime the goal is to have students be active in making sense/reflecting, reading/comprehending text, and comprehending the mathematical structure of a problem situation. If productive language function is a goal, then activity would need to include producing talk or written records to share with peers or the whole class.
b) Any lesson that involves multiple pieces of information through different representations can include MLR4-Information Gap.
c) To support oral discussions, any lesson can include any of the activities in MLR 8-Discussion Supports. Beyond these general routines, in the annotations below, we will suggest other MLRs that are specific to each lesson.

RESOURCE #2: *The ELPD Framework*[18]

This framework describes the multiple language functions associated with the CCSS math practices. Language functions, receptive or productive, are what students do with language to accomplish a math task. .For connections between language functions and mathematical practices, see the Math tables in the ELPD framework (pages 20-25).

As support for the use of best practices in science and mathematics classes, the Council of Chief State School Officers (2012) produced the *Framework for English Language Proficiency Development Standards corresponding to the Common Core State Standards and the Next Generation Science Standards*, commonly known as the *ELPD Framework.* The full document is online at https://ccsso.org/resource-library/english-language-proficiency-development-elpd-framework

RESOURCE #3: *Materials from the Understanding Language project*

Supporting ELLs in Mathematics: The goal of these materials is to illustrate how Common Core aligned math tasks can be used to support math instruction and language development for ELLs at three grade spans (elementary, middle, and high school). Tasks were used or adapted from two publicly accessible curriculum projects, Inside Mathematics and Mathematics Assessment Project. To learn more, see the annotated lessons below and read the Guidelines for Math Instructional Materials Development: https://ul.stanford.edu/resource/supporting-ells-mathematics

"Language of Math" Task Templates: These are language-focused activities that can be used by teachers to design and write their own language-focused activities. These "Language of Math" templates were designed to support students in learning to read and understand word problems, communicate about mathematics, and build disciplinary and academic vocabulary, and develop practices in mathematics. The templates can be found at https://ul.stanford.edu/sites/default/files/resource/2021-02/4.Language%20of%20Math%20Task%20Templates%2010-4-13.pdf

Another version of MLR6-Three Reads titled "Reading and Understanding a Math Problem," a task for supporting academic literacy with word problems, is on pages 37-40 in the document "Language of Math Task Templates available online:" https://meld.sdsu.edu/wp-content/uploads/2020/05/MLR-Presentation-Handouts-4-1.pdf

RESOURCE #4: *Principles for the Design of Mathematics Curricula*

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[18] The ELPD framework was developed by the Council of Chief State School Officers, the English Language Proficiency Development Framework Committee, in collaboration with the Council of Great City Schools, the Understanding Language Initiative at Stanford University, and World-Class Instructional Design and Assessment, with funding support from the Carnegie Corporation of New York.

These principles and accompanying Mathematical Language Routines (MLRs) provide guidance to mathematics teachers for recognizing and supporting students' language development processes in the context of mathematical sense making. The document provides a framework for organizing strategies and special considerations to support students in learning mathematics practices, content, and language. See https://ul.stanford.edu/resource/principles-design-mathematics-curricula

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RESOURCE #5: ***Illustrative Mathematics***

A description of the MLRs and lessons in Illustrative Math courses that include MLRs - "The mathematical language routines were selected because they are effective and practical for simultaneously learning mathematical practices, content, and language. A mathematical language routine is a structured but adaptable format for amplifying, assessing, and developing students' language. The routines emphasize uses of language that is meaningful and purposeful, rather than about just getting answers. These routines can be adapted and incorporated across lessons in each unit to fit the mathematical work wherever there are productive opportunities to support students in using and improving their English and disciplinary language use. These routines facilitate attention to student language in ways that support in-the-moment teacher-, peer-, and self-assessment. The feedback enabled by these routines will help students revise and refine not only the way they organize and communicate their own ideas, but also ask questions to clarify their understandings of others' ideas.

RESOURCE #6: ***MELD Project Materials***

A short three page summary of MLRs and examples from MELD project presentation available here

https://meld.sdsu.edu/wp-content/uploads/2020/05/MLR-Presentation-Handouts-4-1.pdf

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RESOURCE #7: ***Supporting classroom discussions***

Classroom Discussions: Using Math Talk to Help Students Learn, Grades 1-6, Sausalito, CA: Math Solutions Publications by Chapin, S. C., O'Connor, C., & Anderson, N. C. (2003).

Appendix C: Resources and examples of teacher moves to support discussions

1. Chapin, S. C., O'Connor, C., & Anderson, N. C. (2003). Classroom Discussions: Using Math Talk to Help Students Learn, Grades 1-6, Sausalito, CA: Math Solutions Publications. See excerpt here: https://ul.stanford.edu/sites/default/files/resource/2021-02/B.Talk%20Moves%207-2-13.pdf
2. Coggins, D., Kravin, D., Coates, G. D., & Carroll, M. D. (2007). English language learners in the mathematics classroom. Thousand Oaks, CA: Corwin Press (elementary and middle school).
3. Kazemi, E., & Stipek, D. (2009). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Journal of education*, *189*(1-2), 123-137.
4. Kazemi, E., & Hintz, A. (2014). *Intentional talk: How to structure and lead productive mathematical discussions*. Stenhouse Publishers.

Appendix D: Five Practices for Orchestrating Productive Mathematics Discussions

1. **Anticipate**: Actively envision how students might approach the mathematics task they will work on.
2. **Monito**r: Pay close attention to students' mathematical thinking and solution strategies as students work the task, individually or in small groups
3. **Select:** Select particular students to share their work with the rest of the class to get specific mathematics into the open for examination
4. **Sequence**: Make decisions regarding how to sequence the student presentations to maximize connections between and among ideas.
5. **Connect:** Help students draw connections between their solution and other students' solutions as well as the key mathematical ideas in the lesson.

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