# Civic Reasoning and Discourse Across The Curriculum

## MATHEMATICS

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## INTRODUCTION AND OVERVIEW

The National Academy of Education (NAEd) Civic Reasoning and Discourse project aims to support civic reasoning and discourse in the schools by helping teachers across disciplines to prepare youth in the United States to examine and discuss complex issues in ways that are thoughtful, developmentally appropriate, and respectful of others' opinions and experiences. In mathematics, this can mean mathematizing issues that are meaningful and relevant to students' lives—issues that may be related to current curricular topics (e.g., aspects of "fairness") or that are not dealt with in many curricula (e.g., making decisions regarding safe social practices during a pandemic or addressing some of the issues involved in voting rights). Addressing complex topics such as these calls for conceptualizing and representing relevant phenomena; making clear claims; specifying assumptions; supporting justifications with evidence; weighing multiple points of view; and then working through such assumptions, claims, and rationales with others to construct and critique viable arguments.

The curriculum materials that historically have dominated the mathematics textbook market in the United States have provided little support for dealing with current or complex issues, either in terms of content or pedagogy. This report offers such support. It provides background information regarding teaching and learning mathematics, which undergirds powerful classrooms—classrooms from which students emerge as agentive and mathematically, epistemologically, and socially powerful thinkers. It offers a set of principles for powerful pedagogy and practice in mathematics, describing what teachers can offer to ensure that every student is enfranchised and supported equitably in thinking, learning, and acting in the world. It briefly describes three curricular strands connected to current civic issues (fairness, environmental justice, and social justice) that can be addressed with increasing developmental, social, and mathematical sophistication through the grades. It identifies specific developmental considerations for students at the elementary, middle, and secondary grade levels. It then focuses on specifics related to students' mathematical thinking, development, and learning, along with annotated curricular examples for each of the three strands.

Achieving the vision in this document will require facing ongoing challenges regarding curricula and teacher support. Traditional curricula have focused on abstract mathematical ideas, with scant attention to the kind of sensemaking that is internal (seeking and exploiting mathematical coherence) or external (making use of real-world phenomena to give meaning to mathematical objects and using mathematics to model such phenomena in meaningful ways) to mathematics, or that is personal (connecting to previous experiences, understanding, and knowledge) or social (connecting to the collective sensemaking in a particular classroom).

Within mathematics, for example, elementary school students should know that if $3 + 4 = 7$, then they can "undo" the addition and see that $3 = 7 - 4$ and $4 = 7 - 3$; likewise, they know that doubling can be undone by dividing by 2. They should develop the kind of number sense that tells them that the product of 26 and 39 should be close to 1,000 because the product of 26 and 39 should be close to the product of 25 and 40, which equals 1,000. They should know how things grow, so that they can say that the perimeter of a circle (or a pizza that the circle represents) is doubled when the radius is doubled, but the area (or total amount of pizza) is quadrupled. And, they should be able to explain their reasoning, which is an important aspect of sensemaking. In high school, students should be able to explain why the graphs of $y = 2x + 5$ and $y = 3x - 2$ *must* cross, without doing the computation.

External to mathematics, students should learn to make meaningful links between real-world phenomena and their mathematical representations. They should understand, for example, how two graphs represent the costs of two cell phone plans and what the implications are for decision making, what a change in the minimum wage means in terms of purchasing power over the course of a year, or how interest rates on various purchases mount up. Similarly, they should understand how to see the implications of graphs and other mathematical representations that explore the relationships between, for example, real estate values and social services.

Students will need help in learning to do this kind of mathematical sensemaking, and teachers will need support in crafting the kinds of classrooms that help students—often collaboratively—engage in these kinds of activities.

The final section of the report addresses next steps. It offers suggestions for professional development to help make these ideas a reality, and thoughts about possible research and development.

### Framing the Report

In contrast to science, where new ideas (e.g., re-combinant DNA and clustered regularly inter-spaced short palindromic repeats [CRISPR] technologies) quickly enter the curriculum, the mathematics curriculum has been static for decades. Consequently, students often find mathematics less interesting and relevant to their lives than it could be. We hope to support and encourage increased curricular opportunities for student engagement with meaningful and mathematically rich tasks and contexts. This report focuses on the unique challenges of providing opportunities for and supporting civic reasoning and discourse in K–12 mathematics classrooms.

### Essentially, meaningful mathematics learning needs the following:

1. Examples in the curriculum that are relevant to students. Although mathematics curricula include some "real-world" examples to exemplify applications, these examples are typically simplified to the point where they make few meaningful connections to the phenomena that people experience outside the classroom.
2. Less emphasis on "answer-getting." Classroom reasoning often focuses on "answer-getting," using demonstrated procedures rather than carefully thinking through the connections underlying situations.
3. More opportunities for multimodal communication. It is possible to engage students in much more classroom discourse and communication of all types, including talking, listening, reading, and writing. Students should have multiple and frequent opportunities to draft and revise their communications. This is especially important when developing meaningful models of real-world phenomena. Doing so calls for determining the relevance of various issues to the question at hand, understanding the possibility of more than one point of view, and carefully sorting through alternative approaches to problems.

Addressing these needs will enrich mathematics curriculum and instruction.

Mathematics curricula in the United States have evolved slowly over the 20th and 21st centuries, but for the most part they have remained true to their foundations. Instruction at the elementary level historically has focused on the mastery of basic operations, measurement, and the properties of geometric objects. (One superintendent's instructions to teachers in the 1980s said that mathematics instruction should cover no more than the addition, subtraction, division, and multiplication of whole numbers, so that students would be prepared to serve as store clerks [L. Resnick, personal communication, January 10, 1987].) In recent years, the elementary curriculum has opened up to include a greater emphasis on the conceptual underpinnings of fractions and ratios, as well as some descriptive statistics and discussions of statistical distributions. Similarly, curricula at the middle and secondary levels have changed over the past 50 years, both by including statistics and by condensing courses to allow students to study calculus in high school rather than in college. Yet, the standard course sequences and content remain largely the same: Preparation for Algebra is followed by Algebra I, Geometry, Algebra II, Pre-calculus, and then Calculus. Examples tend not to be meaningful in terms of students' personal lives, and instruction often focuses more on implementing procedures than on individual or collective sensemaking.

Much of the power of mathematics resides in its ability to model meaningful real-world phenomena, enabling students to document and discuss issues such as those related to social justice, environmental justice, and fairness. Yet, curricula have typically skirted such issues (Burkhardt & Schoenfeld, 2022), and "applications" are often obviously contrived. The result is that many students see little relevance of or have any interest in mathematics, and others are alienated from it (Gutiérrez, 2013). Moreover, this kind of curriculum results in learners believing that doing mathematics involves producing the answers to exercises or problems in the textbook rather than developing tools for understanding, describing, and changing the world around them.

Significant opportunities exist to expand the mathematics curriculum at all grade levels to include tasks that call for sensemaking and mathematizing, relate meaningfully to students' lives, use mathematically reasonable representations of the issues involved, and help learners understand how they might use mathematics to improve the world around them. Similarly, it would be helpful to put norms and practices in place that make the mathematics classroom a more inviting place in which students feel comfortable exploring and communicating ideas. This means making it safe for students to take risks as they work through ideas. Doing so has the potential to re-frame mathematics as a domain that, while still beautiful and powerful, can be seen as personally meaningful, purposeful, and accessible. It would make mathematics instruction much more consonant with the vision of civic reasoning and discourse presented in the 2021 NAEd report *Educating for Civic Reasoning and Discourse* (Lee et al., 2021), in which students learn across the curriculum to address meaningful challenges in thoughtful, analytic, and respectful ways.

## CRAFTING MATHEMATICALLY POWERFUL CLASSROOMS: PRINCIPLES

This section begins with some general principles about learning, reflecting the breadth of research on learning and development in the introduction to the practitioner report series. We then transition to the mathematics classroom, reflecting on how these broad principles need to be considered when teaching. We describe the principles underlying powerful mathematics learning classrooms—classrooms from which all students emerge as powerful and empowered mathematical thinkers and learners.

### Overarching Principles

Human learning is universal and follows certain general patterns (see the *Handbook of the Cultural Foundations of Learning* [Nasir et al., 2020]). Key ideas in the *Handbook* are that learning is rooted in the evolutionary, biological, and neurological systems of our bodies and minds; is inseparable from our social and cultural activities; is not a separate activity but is intimately tied to development, emotion, and the formation of identity; and is inherently a social activity, shaped by everyday cultural activities in and out of school and across the lifespan (see the discussion of the RISE framework in Nasir et al. [2021]). These ideas apply to all learning, including the tacit reference to communication in the claim that learning is inseparable from our social and cultural activities. It is essential to keep these ideas in mind, even when we think about what takes place in classrooms.

The goal is for learning environments to be consonant with the general principles discussed above (Seda & Brown, 2021). Hence, classrooms should function as learning communities in which everyone contributes to knowledge generation, not just the teacher. In addition, such learning communities depend on and draw from knowledge outside the classroom. They build critical awareness and reflection, and they support learners' active thinking and problem-solving. Students become independent thinkers in these environments through culturally relevant and responsive curricula that build on students' assets—languages, cultural practices, identities, and prior knowledge and experiences (see Figure 1 for the ICUCARE Framework). Too often, issues like tracking, low expectations, and staffing problems impact the ways some students (particularly low-income students, students of color, and multilingual students) are able to engage in high-quality mathematics instruction (Seda & Brown, 2021). An equity framework like ICUCARE can serve as a lens for helping educators to evaluate and improve their teaching practices to be more inclusive of all of their students.

### Inside the Classroom: The Essential Dimensions of Mathematically Powerful Classrooms

A fundamental goal of mathematics instruction is to provide the kind of support that enables every student to emerge as an agentive and mathematically, epistemologically, and socially powerful thinker. Thus, issues of how students' mathematical identities are developed, manifested, and supported are central.

Literature reveals multiple aspects of identity and its shaping in social contexts (e.g., Greeno, 2006; Holland et al., 1998; Koole, 2003; Turner et al., 2013; Wenger, 1998). Identities are shaped in multiple ways—by membership in communities (e.g., "I am a reader; I am a member of a great book club"), by stereotypes (e.g., certain groups are considered good at math; certain other groups are not),

***Figure 1:*** **The ICUCARE Framework**

| | |
|---|---|
| Include others as experts | Create classroom environments that extend beyond the teacher as the sole authority to develop competence and confidence in others as experts, including the students themselves. |
| Be Critically Conscious | Take the time to understand how negative stereotypes impact our students and actively work to erase the effects of those negative stereotypes on the educational outcomes of diverse learners. |
| Understand your students well | Learn about your students, their families and their communities for the purpose of improving instruction. *(Not making assumptions)* |
| Use Culturally relevant curricula | Use instructional materials in ways that help students see themselves as doers of mathematics and help them to overcome the stereotypes and messages regarding who is mathematically smart. |
| Assess, Activate and build on prior knowledge | Value the prior knowledge that students bring to the classroom, both personal and cultural, and use the knowledge as a resource for creating new knowledge. |
| Release control | Empower your students to take ownership of their own learning by focusing on sensemaking and allow them to make choices about things that are important to them in the classroom. |
| Expect more | Hold high expectations for all sstudents and avoid deficit views of diverse learners. |

*Note.* Seda and Brown (2021), with permission.

and by the ways people position each other either explicitly (e.g., "Let's ask José. I bet he knows") or tacitly (e.g., when a group ignores a contribution made by one student but listens when another student with perceived higher social status makes the same comment).

Identity is closely related to agency, a willingness to take on tasks and contribute either individually or as part of a group effort (see Aguirre et al., 2013). When students actively engage in and contribute to mathematical (or other) discussions, it can be inferred that they have productive mathematical (or other) identities. Students with *conceptual agency* want to understand and make connections (Cobb et al., 2009). Furthermore, *critical mathematical agency* expands on the notion of agency to include the idea of engaging in mathematical activities that are both personally and socially meaningful (Turner, 2003). This is a key goal of instruction that helps to prepare students for engagement in civic discourse.

As we have noted, agency and identity are constructed through social interactions. In classrooms that provide *all* students with opportunities to explain and justify solution strategies, pose questions, and articulate connections between mathematical ideas—in ways that are personally safe—students have opportunities to become (more) mathematically agentive and build (more) productive mathematical identities.

Mathematical agency and positive mathematical identities are goals for every student—this is a fundamental issue of equity. The question is how to get there. That is, how can the important aspects of learning environments be organized so that they can be understood and worked on for purposes of continuous improvement? The Teaching for Robust Understanding (TRU) Framework organizes and distills the extensive literature on teaching into five essential dimensions of classroom practice (see Figure 2).

*Figure 2:* **The Five Dimensions of Powerful Mathematics Classrooms**

| The Five Dimensions of Powerful Mathematics Classrooms | | | | |
|---|---|---|---|---|
| **The Mathematics** | **Coginitive Demand** | **Equitable Access to Content** | **Agency, Ownership, and Identity** | **Formative Assessment** |
| The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful mathematical thinkers. Discussions are focused and coherent, providing oppourtinties to learn mathematical ideas, techniques and perspectives, make connections, and develop productive mathematical habits of mind. | The extent to which students have opportunities to grapple with and make sense of important mathematical ideas and their use. Students learn best when they are challeneged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conducive to what has been called "productive struggle." | The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematical content being addressed by the calss. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how richm the content: all students need to be involved in meaningful ways. | The extent to which students are provided opportunities to "walk the walk and talk the talk" - to contribute to conversations about mathematical ideas, to build on others' ideas and have others build on theirs - in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners. | The extend to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets student where they are" and gives them opportunities to deepen their understandings. |

*Note.* Figure developed by Alan Schoenfeld and adapted from Schoenfeld (2013, 2014, 2022) and Schoenfeld et al. (2023a,b). See https://truframework.org/.

This research leads to a series of questions that teachers can ask during planning, teaching, and reflection:

- What mathematics should these students (at a particular age, given developmental considerations, etc.) experience and in what ways? (Dimension 1: The Mathematics)
- How can instruction be adjusted in ways that position these students to build on prior knowledge and expand their knowledge base with new ideas? (Dimension 2: Cognitive Demand)
- How can instruction be organized so that every student in a classroom, school, or district participates meaningfully with regard to the content, engaging in relevant mathematical practices? (e.g., those in the National Council of Teachers of Mathematics [NCTM] and Common Core State Standards; Dimension 3: Equitable Access to Content)
- How does the content relate to students and their personal lives? How does their engagement in classroom discussions provide opportunities to get their ideas heard and discussed in ways that are respectful and enfranchising? (Dimension 4: Agency, Ownership, and Identity; note the essential connection to civic discourse)
- How is student thinking made available for collective discourse and productive feedback, whether from the teacher or other students? (Dimension 5: Formative Assessment)

Before expanding on the TRU Framework, the next section addresses the central role that language plays in supporting powerful and empowering classrooms. Then the report will offer a collection of practices, organized by dimension, that helps teachers craft mathematically powerful classrooms.

### Attending to Language

Using oral or written language is central in each of the five dimensions listed above. Students bring to the classroom their own meanings for the language they use; they also bring ways of talking that fit outside the classroom (e.g., slang, jokes, how to present evidence). In the classroom, teachers use mathematical terms and phrases with very specific meanings as well as mathematical ways of talking or writing (e.g., conjecturing, justifying, generalizing, referring to visuals). Ultimately, students learn how to think and communicate using those terms and phrases with mathematical meanings and how to use those valued ways of communicating—in other words, "the mathematical register." The question is how to bridge the gap by building on the everyday register and students' own sensemaking processes (Moschkovich, 2012, 2014).

The main language issues in teaching mathematics are hearing the math in students' contributions, even when the ways the math is expressed may be imperfect or imprecise; supporting students in communicating mathematical ideas through language using informal and formal registers; and connecting students' ways of talking to more formal registers. Learning to communicate using those informal (everyday, home, etc.) and formal (school, mathematical, or academic) registers is part of Dimension 1 in the TRU Framework—the Mathematics. Attention to mathematical language and its relation to everyday language use is essential. At least as important, sensemaking depends on having ideas within reach (Dimension 2: Cognitive Demand) and opportunities to express initial or exploratory ideas through emergent language that is often imperfect (Moschkovich, 2014, 2016). Sensemaking is an activity that fundamentally involves language—not only as a communication tool but also as a thinking tool. If students are not supported in participating in mathematical conversations and written communication, they do not have equitable access to core math ideas and practices (Dimension 3: Equitable Access to Content), and they certainly do not have opportunities to engage in ways that build mathematical identities (Dimension 4: Agency, Ownership, and Identity). Finally, formative assessment (Dimension 5) fundamentally involves speech, text, and other modes of communication (e.g., drawings, graphs, tables, visuals).

Math understanding and language development are interconnected and mutually reinforcing. Instruction that attends to language needs to focus on what students are using language for rather than the exact way that students say, record, or write their contributions. That means doing more than focusing on vocabulary or simply providing students with the mathematical meanings of terms to use or memorize (such as vocabulary lists, definitions, word walls, etc.). Students need to use language actively to develop and negotiate meanings, not only for single words, phrases, or definitions but also to learn how to express math practices through language (e.g., by learning how to construct a good justification, make a claim more precise, or connect a conjecture to a mathematical representation). As students share their thinking with peers, they can start to refine their language; over time, teachers can support students in moving toward more effective mathematical communication as they revise and refine oral and written expressions of their math work.[1] Box A provides a series of language routines that have been shown to enhance students' participation and understanding in mathematics classes.

Language matters in math classrooms for all students: proficiency in formal or academic mathematical communication is a learned skill in math classrooms, and all students need opportunities to use both informal and formal mathematical language to communicate their ideas as they explore a topic or solution. Thus, many of the teaching strategies that support discussions are useful for attending to language.

[1] For an example of such scaffolding during a classroom discussion, see Moschkovich (2015).

## CRAFTING MATHEMATICALLY POWERFUL CLASSROOMS: PRACTICES

The previous section outlined five fundamental dimensions of mathematics classrooms. Compelling evidence demonstrates that to the degree that classrooms do well along those five dimensions, the students in them will develop into motivated, agentive, and powerful mathematical thinkers.

In this section, we turn from principles to teaching practices—the actions teachers can take to curate mathematically powerful classrooms along the five dimensions of the TRU Framework. For each TRU dimension, we list relevant classroom practices that help to enrich classroom activities.

A

### Mathematics Language Routines

The eight Mathematics Language Routines (MLRs) provide opportunities over extended time for students to participate actively and improve using language to communicate mathematically, both orally and in writing (Daro, 2021; Zwiers et al., 2017). These routines reflect the collective wisdom of many years of research and practice and have been shown to enhance students' participation and understanding in mathematics classes.

Detailed descriptions of the following MLRs can be found at Imagine Learning Classroom[2]:

- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- MLR3: Clarify, Critique, Correct
- MLR4: Information Gap[a]
- MLR5: Co-Craft Questions
- MLR6: Three Reads[b]
- MLR7: Compare and Connect
- MLR8: Discussion Supports[c]

When engaging students in MLRs, it is important to (1) ensure that students are active analytically (e.g., reflecting, problem-solving, sensemaking) and (2) provide opportunities for students to use language in both receptive (e.g., comprehending, listening) and productive (e.g., talking, writing) functions as well as to revise what they have produced. Examples of using MLR in a multilingual secondary classroom are also available (e.g., see Zahner et al., 2021).

[a] Any exemplar lesson in this report that involves multiple pieces of information through different representations can include MLR4: Information Gap.

[b] Any exemplar lesson in this report that involves reading text can include MLR6: Three Reads. This routine is useful to engage students in active sensemaking and reflecting, reading and comprehending text, and understanding the mathematical structure of a problem situation. If productive language function is a goal, then the activity would need to include producing oral or written records to share with peers or the whole class.

[c] To support oral discussions, any exemplar lesson in this report can include any of the activities in MLR8: Discussion Supports.

[2] https://ilclassroom.com/wikis/178268-ell-mathematical-language-routines

## TRU Dimension 1

### Powerful Practices for Crafting Rich Mathematical Environments

The goal of Dimension 1 (The Mathematics) is for mathematics learning to be a rich experience. In addition to being focused on big mathematical ideas, students should have opportunities to engage in key mathematical practices including sensemaking, building connections, reasoning, and explaining.

To achieve this goal, text materials can be enhanced in a variety of ways. A key idea is to keep students focused on sensemaking. This can be done by posing questions for small group discussion as well as in whole class discussions:

- "Does anyone have ideas about how we can get started?" (This and similar questions can open up discussions.)
- "Does the answer seem reasonable?" (If someone's grandmother turns out to be 7 years old, or someone is running at 2 miles per hour, there's a problem!)
- "Does anyone have a different way of thinking about this?" (Such questions can support students in comparing methods and help them make connections, prompt them to share their thinking, and understand each other's way of thinking.)
- "How would you explain this idea (or problem or solution) to a student who missed today's class?" (This makes their peers the audience for their communication.)

Mathematically rich problems can stimulate especially meaningful discussions. Discussion of "group-worthy" problems—problems that can be approached and solved in more than one way—encourages sensemaking and making connections between the insights and methods that students develop (Cohen & Lotan, 1997, 2014; Lotan, 2003).[3]

## TRU Dimension 2

### Powerful Practices Related to Supporting Sensemaking and "Productive Struggle"

The key concepts related to Dimension 2 (Cognitive Demand) are sensemaking and productive struggle. Students should actively build their own understandings as they work. Tasks that are too easy do not help them to grow mathematically, but tasks that are inaccessible do not help either—they just build frustration. Therefore, a significant amount of monitoring is called for (see also Dimension 5: Formative Assessment), as is making adjustments (i.e., scaffolding) so that students are working on tasks with balanced difficulty levels.

As noted above, using mathematically rich tasks—tasks with multiple entry points and multiple solutions—allows more ways for students to engage in a task and then to compare and contrast their thinking with others. Teachers cannot adjust tasks or make suggestions productively unless student thinking is observable; this is another reason for collaborative group work, whole class discussions, public records, and student writing. Also, it is difficult for students to revise their thinking unless it is recorded in some way. Equally important is the construction of classroom norms in which students can make mistakes, reflect on them, and then make progress—building up solutions to complex problems is a gradual process.[4]

[3] For a collection of 100 lessons that include mathematically rich tasks and extended lesson plans to scaffold productive classroom dialogue, see the formative assessment lessons at https://www.map.mathshell.org/lessons.php

[4] See Box A for some of the routines that attend to both mathematics and language, including "Three Reads"; "Stronger and Clearer Each Time"; and "Clarify, Critique, Correct."

## Powerful Practices to Support Each Student's Mathematical Agency, Ownership of Ideas, and Identity

The goal for mathematics instruction is that each student becomes mathematically powerful, developing a sense of mathematical agency and a positive mathematical identity. For that reason, we discuss teaching practices related to Dimensions 3 (Equitable Access to Content) and 4 (Agency, Ownership, and Identity) of the TRU Framework together. Equitable access to central mathematical content and practices is crucial—and the right kind of access should contribute to the development of positive mathematics identities.

The key issue with regard to equitable access is to make sure that *every student engages with the core mathematical content of the lesson*, whether in whole class or small group activities. It is easy for some students to take over and let others ride on their coattails, or for some students to hide. Maintaining a classroom where all students feel free and encouraged to participate in matters of mathematical import can be challenging. Not all students will participate in the same way; for example, some students may choose writing over talking, and others may talk more in small groups than in whole class discussions. But, having some students practice "basics" while others engage in meaningful problem-solving ultimately creates inequities and increases the performance differences between the two groups. Students are not likely to learn if they do not have the opportunity to engage in problem-solving, reasoning, or other key math practices.

Some equity-focused practices include emphasizing sensemaking over speed, discussing and highlighting various ways of being mathematically smart (generating ideas, organizing a group's work, serving as sounding board and questioner, recording and summarizing a group's ideas, etc.), and highlighting and building on student contributions so that reluctant sharers see their ideas as welcome. This too takes practice. Instead of looking for and certifying the "right" method for approaching a particular problem, teachers could look for the productive elements in student thinking, with the goal of working them into classroom conversations. Student contributions might include ideas, connections, conjectures, explanations, etc. Teachers can "revoice" student contributions, perhaps clarifying or paraphrasing but giving the students credit for the ideas.

Some teachers use randomizing devices such as equity sticks. When used in nonthreatening ways (e.g., so that opting out can be done safely), such randomizing devices can create expectations of participation and distribute opportunities to participate widely. Similarly, students can be invited to use a wide range of resources that support mathematical thinking: everyday and home language; culturally grounded or invented algorithms (which can be compared to standard algorithms); and an array of representations including drawings, gestures, objects, and symbols. Table 1 includes a collection of powerful identity-related practices.

**Table 1** Five Equity-Based Mathematics Teaching Practices

| Practice | General Actions | Specific Practices |
| --- | --- | --- |
| Go deep with mathematics | Implement tasks that promote reasoning | ■ Support students in justifying claims<br>■ Present high cognitive demand tasks<br>■ Implement multiple solution strategies and representations |
| Leverage multiple mathematics competencies | Structure collaboration to use a variety of skills, knowledge, and levels of confidence | ■ Support students in contributing in different viable ways<br>■ Present tasks with multiple ways to enter<br>■ Encourage multiple ways to communicate<br>■ Create space to engage students with different levels of confidence, skills, and knowledge |
| Affirm mathematics identities | Support productive struggle | ■ Promote student persistence and reasoning<br>■ Support students in seeing themselves as making valuable contributions<br>■ Assume mistakes are learning opportunities<br>■ Validate students as math learners<br>■ Focus on student competencies and recognize various ways to show competence |
| Challenge spaces of marginality | Position students as sources of expertise for solving complex problems | ■ Generate questions<br>■ Distribute mathematics authority among students, teachers, and mathematics text<br>■ Encourage student-to-student interaction<br>■ Encourage public reasoning and discussion |
| Draw on multiple sources of knowledge | Use and connect mathematical representations; elicit and use evidence of students' thinking | ■ Make intentional connections<br>■ Bridge previous learning to new learning<br>■ Tap mathematical knowledge from community, family, culture, etc.<br>■ Recognize and strengthen multiple language forms<br>■ Affirm and support multilingualism |

*Note.* Adapted from Aguirre et al. (2013, pp. 45–48).

## TRU Dimension 5

### Using Formative Assessment to Enhance Student Learning

The key to formative assessment is gathering information about student understandings *prior to and during instruction* that can be used to adjust instruction so that it meets the students where they are—for example, supporting rich mathematics, identifying false starts and misconceptions (and dealing with them carefully, not simply making corrections), making sure the level of cognitive demand is right, and making sure students have meaningful opportunities to have their voices heard.

Giving pre-assessments (assignments that reveal what students know and do not know before diving deeper into a unit or lesson) allows teachers to learn the knowledge, skills, and dispositions students already have and to build on them. Similarly, using tasks that reveal misunderstandings, and having classroom discussions of them, allows students to work through their ideas in ways that are more powerful than simply saying what is right or wrong.

Providing students with multiple ways to show their thinking—not only orally but also in writing (which allows revising), and not only as part of the whole class or individually but also in small groups—supports equitable access to feedback from multiple sources. It is worth noting that having students collaborate makes life easier for teachers. When the burden of diagnosis and remediation falls solely on the teacher's shoulders, formative assessment is a nearly impossible challenge. But, when students work according to the norms discussed above (asking questions, asking for justifications, etc.), many learning issues get resolved in small groups or pairs before the teacher even learns about them.[5]

## EXPANDING THE CURRICULUM: A RANGE OF ISSUES THAT CAN BE MATHEMATIZED

Current mathematics curricula tend not to provide students with significant opportunities for modeling, decision making, and the kinds of civic discourse needed to address complex and relevant "real-world" problems. Discussions of these topics can provide rich opportunities for learning to engage in civic discourse and using mathematics as a tool for reasoning and argumentation. The question is as follows: What bodies of work, or approaches, might be useful for students to experience? Several possibilities offer mathematical richness and support civic discourse (see Box B for a range of topics that might be explored). Additionally, developing critical literacy skills facilitates meaningful engagement in mathematics classrooms (see Box C for a discussion of critical literacy skills within the context of mathematics instruction).

[5] The *100 Classroom Challenges* lessons (https://www.map.mathshell.org/lessons.php) provide a large number of classroom activities and support materials in formative assessment for middle and secondary school mathematics, including extensive lesson plans and examples of supportive discourse practices.

B

## Topic Ideas: Opportunities for Civic Reasoning and Discourse

**Issues concerning public health.** Policies addressing public health issues can seem arbitrary, and they may go unquestioned or ignored—but students can reason through underlying data and policy questions. For example, several social issues are related to COVID-19—masking, social distancing, and how COVID-19 infection rates correspond to race or ethnicity and socioeconomic status. To consider issues of vaccination, for example, teachers can ask guiding questions such as the following:

- What data would you want to gather to see if vaccines have an impact?
- How would you analyze those data?
- What is the downside of vaccination?
- What data would you want to gather to explore the risks of vaccination?
- How would you analyze those data?

**Issues concerning macro-level demographics.** Students can explore data related to population growth and resource scarcity, and learn to represent the data mathematically. The Population Reference Bureau (prb.org) put together a World Population Data Sheet focusing on the census throughout history.

**Issues concerning electoral participation.** Teachers can find many pointers to lessons by doing an internet search on “gerrymandering curriculum tasks.” Games and other tools can be found by searching for “gerrymander simulations” or “gerrymandering example.” High School Mathematics Lessons to Explore, Understand, and Respond to Social Injustice[6] (Berry et al., 2020) includes a gerrymandering task. The example of “fairness” illustrated in the “Fairness as a Context for Mathematics Learning” section below can be modified for the elementary classroom, where Xs and Ys can correspond to boys and girls. (See figure 3A and 3B, page 17)

C

## Critical Literacy Skills

Developing students’ critical literacy skills can enhance engagement and understanding in mathematics classrooms (Burkhardt & Schoenfeld, 2022). Stated another way, how does one take any current issue that has mathematical underpinnings and make sense of it? This includes critically reading, interpreting, and analyzing data in a situation to figure out what information is needed, what claims can be made, what makes a claim clear and precise, what are sources of strong evidence, etc. As the phrase “lies, damned lies, and statistics” that famously critiques applied statistics indicates, it is essential for students to understand what it means to draw fair conclusions from a body of data and how data can be misrepresented. This may sound very advanced for elementary school students. However, these points can be illustrated with relevant examples (e.g., giving one student a large cookie and one student a small cookie and claiming that the distribution is fair because each student got the same number of cookies, or giving each family the same amount of money for holiday gifts when some families have one child and others have more).

[6] https://www.amazon.com/School-Mathematics-Lessons-Explore-Corwin/dp/154435259X

## Three Key Themes: Fairness, Environmental Justice, and Social Justice

There are many possible ways of expanding the mathematics curriculum to include topics that are mathematically rich and that have the potential to support rich civic discourse. In this section, we briefly describe topics or contexts under the three themes of fairness, environmental justice, and social justice that can be used to engage students with rich mathematical content and to support thoughtful and respectful civic discourse across the K–12 spectrum (and beyond). Through sample lessons in the following "Elementary School," "Middle School," and "Secondary School" sections, we provide and annotate one example from each theme at each school level, suggesting how the tasks can be considered for and used in the classroom.

This report focuses on these three themes for two significant reasons: (1) in addition to being mathematically rich, these topics have the potential to be personally meaningful for students; and (2) the topics can be approached in various ways, depending in part on what aspects of the topics teachers and/or students consider to be important. We emphasize that the role of mathematics is to help students understand complex situations in which strong opinions often are only partly supported by evidence. Ultimately, students should develop dispositions and skills that enable them to think carefully about important social issues that can be illuminated by mathematics—for example, if they see claims in the media about some form of inequity, they can frame inquiries in neutral ways, decide which factors or variables are relevant, consider what data would inform the issue, and find and analyze those data.

Addressing potentially controversial issues calls for empathy and sensitivity in both personal and interpersonal terms. Some discussions may be painful for some students—for example, a discussion of COVID-19-related issues may bring to mind the pain and suffering of relatives who contracted the infection and may have died from it. Other topics may raise issues of personal identity or group affiliation that may lead to stereotyping, which is typically not seen in classrooms that focus narrowly on conceptual mathematical topics. Moreover, the background knowledge required for addressing such issues is broader than the knowledge required for teaching straightforward mathematics topics.

For that reason, each of the sample lessons described in the following "Elementary School," "Middle School," and "Secondary School" sections will include brief annotations related to the following components of civic reasoning and discourse: knowledge, epistemology, dispositions, ethics, language, and developmental needs (see Box D for more detailed descriptions).

Delicate political context should also be considered when addressing issues such as fairness, environmental justice, and social justice in the classroom. Indeed, in states that have banned the teaching of topics such as the enduring legacies of slavery, the very mention of justice may be seen as problematic. Consequently, some teachers or districts may decide that one or more topics we have chosen to exemplify are "too hot to handle" in the classroom. Our stance is that relevant resources must be widely available, whether or not a particular school or district will choose to use them. One reason we highlighted specific possible topics (see Box B) and critical literacy skills (see Box C) is to emphasize the fact that schools and districts have many topics to choose from when they think about opening up mathematics lessons to modeling issues suitable for civic discourse. The discussions below indicate the ways in which such topics can be approached.

### *Fairness as a Context for Mathematics Learning*

Fairness is a well-studied area of the mathematics curriculum. Issues of fairness undergird much of the curriculum at all grade levels. For example, fractions represent "fair shares" of some quantities divided equally among a group of people. How do you share two different-sized cookies fairly between two students? Or a snack-size box of raisins? Things can get complex rapidly. Suppose four people want to share three cookies, and the cookies are not the same size. What about cutting a square frosted cake so that three people each get the same amount of cake and the same amount of frosting?

**D**

## Key Components of Civic Reasoning and Discourse in K–12 Mathematics Classrooms

- Knowledge: the underlying conceptual knowledge and the skills required to address the civic-related issue at the core of each lesson (at each grade level).
- Epistemology: the questions of what constitutes mathematical knowledge; what evidence is used to evaluate claims, including how an issue is framed; and what assumptions are made.
- Dispositions: the willingness to approach challenging issues while putting preconceptions aside, to weigh evidence fairly, to engage in respectful discourse about sensitive topics, and to develop the ability to engage in thoughtful questioning.
- Ethics: the recognition that some alternative considerations and possible approaches to situations are based on values, that there is not necessarily one answer to complex situations, and that there is a need to search for solutions that are right and just.
- Language: the understanding that oral, text, and visual displays are central aspects of learning to reason mathematically, and that the linguistic resources students bring to the classroom can serve as a basis for developing and refining their understandings regarding mathematical situations and communicating about them (see Box A).
- Developmental needs: the biological, cognitive, and psychosocial development of children and adolescents as acknowledged and fostered in supportive, engaging classroom environments.

Or, suppose students want to share the costs of carpooling to school, but each participant lives a different distance from school. How do you think about allocating costs in that case? If the students were riding a bus, they would all pay the same fare; yet, in some transit systems, they would be paying more for longer trips.[7]

Then there are issues of fairness in representation and apportionment. In Figure 3A, which shows five voting districts, Ys are 40% of the population and win two out of five, or 40%, of the five vertical districts. In Figure 3B, Ys are 40% of the population and win 0% of the five horizontal districts. What's "right" is not so simple anymore.

***Figure 3A:* The Ys Win 40% of the Districts**

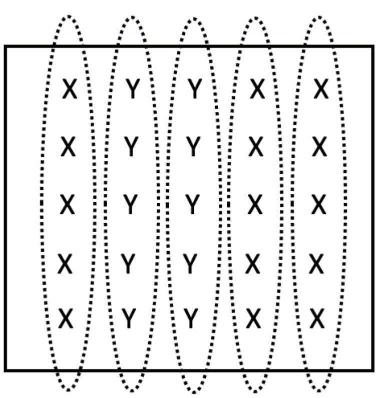
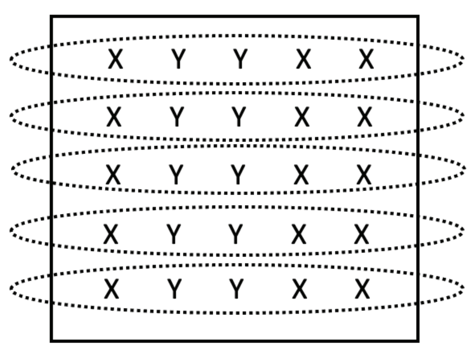



***Figure 3B*: The Ys Lose in Every District**




*Note.* Adapted from https://www.choices.edu/wp-content/uploads/2018/10/choices-twtn-gerrymandering-info-1.pdf

[7] For a unit that explores this issue, see https://www.map.mathshell.org/lessons.php?

Issues of representation and apportionment in the real world are much more complex as well as potentially controversial. Furthermore, while mathematical tools provide one way of determining fairness when agreed-upon "equal shares" or some other form of "equal" is what is considered collectively to be fair, there are also perspectives that do not rest on notions of equality in terms of what counts as fair. For example, if the goal is for every student to have a clear view of the stage during a performance, some students may need extra pillows. The unequal distribution of pillows turns out to be fair.[8]

### *Environmental Justice as a Context for Mathematics Learning*

Major climate disasters are occurring with increased frequency. "Once in a century" firestorms now rage on a regular basis, torrential rains and flooding occur across the globe, rising sea levels threaten whole cities and coastlines, and increasing temperatures are making some areas unlivable. At the same time that rain forests are being burned to the ground, laws or policies have been enacted that appear to favor one part of the energy market over another (e.g., gas-guzzling trucks are exempted from regulations). Environmental justice is also an urban issue. Environmental injustice results from the interaction of economic disparity, social inequity, and unequal access to environmental or health resources. Many students from communities of color and low-income groups reside and study within expansive urban districts, where they bear the brunt of environmental injustice in glaring disproportion. See Box E for examples of environmental justice-related curricular materials.

**E**

## Materials for Environmental Justice–Related Assignments and Projects Across Grade Levels

There is an emerging body of curricular materials related to environmental justice. Many lessons are found in broader collections related to social justice. Below are some examples:

- "How Air Pollution Across America Reflects Racist Policy From the 1930s"[9] (The New York Times) illustrates how redlining contributed to air pollution across America
- "Redlining means 45 million Americans are breathing dirtier air, 50 years after it ended"[10] (The Washington Post) addresses how redlining has created pollution disparities in more than 200 U.S. cities
- Very broad coverage is given in publications such as A People's Curriculum for the Earth: Teaching Climate Change and the Environmental Crisis[11]
- NCTM resources[12]
- Eco-Mathematics Education: K-8 Lesson Plans for Ecological and Social Change[13](Brill)
- "Sustainability Math"[14]

[8] An internet search on individual topics yields a wide range of resources, including voting and redistricting simulations. A collection of curricular resources can be found at https://www.nctm.org/Search/?query=fairness#?wst=d96f7c2823ec8a70319bf3c5b2c7fcd5
[9] https://www.nytimes.com/2022/03/09/climate/redlining-racism-air-pollution.html
[10] https://www.washingtonpost.com/climate-environment/2022/03/09/redlining-pollution-environmental-justice/
[11] https://rethinkingschools.org/books/a-people-s-curriculum-for-the-earth/
[12] https://www.nctm.org/Search/?query=environmental%20justice#?wst=d8a3428bd9d5122a0589af75d93b2c33
[13] https://brill.com/display/book/9789004466807/BP000007.xml?language=en
[14] https://sustainabilitymath.org/ https://rethinkingschools.org/

F

### Social Justice–Related Learning Materials

There is a small but robust collection of materials that can be used as resources in this arena:

- Rethinking Schools[15] has a large collection of relevant materials, specifically those in Rethinking Mathematics[16].
- A series of books published by Corwin (Bartell et al., 2023; Berry et al., 2020; Conway et al., 2022), from which this report draws several lesson examples, provides lessons at the elementary, middle, and secondary levels.
- NCTM offers a wide variety of resources[17], including the Catalyzing Change series[18].

## *Social Justice as a Context for Mathematics Learning*

Social justice issues—that is, any issues for which there might be disparities in treatment or status of different groups—can be addressed with increasing sophistication from the early grades through college. Students can investigate issues and engage in reasoning around questions such as the following: who gets different kinds of health services? What are the COVID-19 infection rates and infant mortality rates for different subgroups of the population? Such learning can be embedded in mathematic lessons or student projects. See Box F for examples of social justice-related learning materials.

15 https://rethinkingschools.org/
16 https://rethinkingschools.org/books/rethinking-mathematics-second-edition/
17 https://www.nctm.org/socialjustice/
18 https://www.nctm.org/change/www

## Issues and Challenges Particular to Elementary School[19] Mathematics

The contexts of and conditions for learning differ at every grade band. As students grow, their memory, capacity for abstraction, social skills, and personal interests can change dramatically. At the same time, the curriculum changes: it becomes more abstract and, in general, more fast-paced. Dealing with all these issues is a major challenge. Here we list some of the challenges particular to elementary school.

Mathematics is and should be experienced as a socially negotiated sensemaking activity from children's earliest encounters with it. Young children are naturally inquisitive. They notice patterns; invent ways of talking about quantities, relationships, changes, and space; and develop representations of these. These developments often go unnoticed by adults. Young children should be encouraged to explore and be supported as they discover and reason with and about mathematical ideas. Communicating their understandings and refining them in interaction with others is essential (Lampert, 1990; Lampert et al., 1996).

In the early grades especially, mathematics learning should be about sensemaking. Children are discovering who they are. They are putting together their sense of self, and they are learning to use language. They are learning their home language, and some may also be learning English as an additional language. In school, they are learning to use multiple modes to communicate their thinking through speech, writing, drawings, graphs, tables, visuals, etc. They are also starting to learn how to communicate in more formal and mathematical ways. They are beginning to see and understand other people, and to see themselves reflected in how others see and treat them. For all these reasons, classroom discourse must reflect the kinds of considerate, respectful, thoughtful, and reflective discourse we hope students will engage in, in all arenas, throughout their lives.

[19] Here we use the phrase "elementary mathematics" to refer to mathematics in grades K–5 and "upper elementary" to refer to grades 3–5.

## LESSON A: FAIRNESS

In the lesson "Tu Lucha Es Mi Lucha: Mathematics for Movement Building," students are introduced to decimal concepts and operations as they explore the salaries of the migrant farmworkers who went on strike to raise their pay in the 1960s.[20] Students use choral counting based on the hourly wage of agricultural workers and use the results to solve a real-world problem related to workers' living conditions. This lesson raises issues of fairness and its mathematical documentation in a historical context—but in ways that are directly related to contemporary issues as well as what it means to earn a living wage. The content is aligned to the fifth grade California Mathematics Standards (related to operations on decimals, concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction). Also, students use the number line to represent fraction (or decimal) magnitudes and operations. They begin to interpret data distributions to answer questions and pose further questions.

In terms of relevance, the authors of the lesson live and work in California, a state with a very large migrant worker population. Many children in the state have family members who work in agriculture. California celebrates Cesar Chavez Day, which provides the perfect context for the problem. Of course, the issues of fair pay addressed in this lesson are universal. The lesson is developmentally appropriate, as fairness is a concept that children understand at an early age (Turiel, 2006, 2015).

### Lesson Outline.

#### *Launch.*

In a quick free write, students are asked to respond to the following question in their journals:
*How would you feel if your pay rate was based upon your race, ethnicity, or gender?*

- The students do a "think, pair, share" with a partner, and a whole group share-out.
- The students look at images of grapes and do a "notice and wonder."
- The students answer questions about their knowledge of migrant farm work.
- The students are shown a video about Larry Itliong, a Filipinx farmworker and organizer of the United Farm Workers Union.[21]
- The students are asked to respond verbally to the following question:
  *What are you wondering about after viewing the video?*

**Table 2** Example of a Choral Counting Table

| | | | | | |
|---|---|---|---|---|---|
| 1.25 | 2.50 | 3.75 | 5.00 | 6.25 | 7.50 |
| 8.75 | 10.00 | 11.25 | 12.50 | 13.75 | 15.00 |
| 16.25 | 17.50 | 18.75 | 20.00 | 21.25 | 22.50 |
| 23.75 | 25.00 | 26.25 | 27.50 | 28/75 | 30.00 |
| 31.25 | 32.50 | 33.75 | 35.00 | 36.25 | 37.50 |
| 38.75 | 40.00 | 41.25 | 42.50 | 43.75 | 45.00 |

[20] The complete lesson can be found in Bartell et al. (2023).
[21] See https://ca.pbslearningmedia.org/resource/filipino-american-farmworkers/asian-americans-video/

***Explore.***

Filipinx and Latinx laborers worked in separate fields and lived in separate camps. The Latinx laborers were paid $1.40 per hour, but the Filipinx laborers were paid $1.25 per hour. Students explore the Filipinx hourly wage, with the goal of addressing the following question: *Is this fair?* The students engage in a choral count, starting with $1.25 and adding $1.25 each time. The teacher arranges the numbers in an array that has six columns in order for students to identify the amount paid for 12 hours of work—one day's pay—more easily (see Table 2).

The students spend time looking for patterns in the table and sharing their "noticings" and "wonderings." After discussing patterns in the table, the students are given the following problem to work on:

> In the early 1960s, manongs, Filipinx farmworkers, were paid $1.25 per hour of labor. On average, farmworkers were expected to work 12 hours a day, 6 days a week. How much did the manongs earn in one day? The manongs had little money and relied on low-income housing, for which rent was $50 a month.
>
> How much would the manongs make in a month? How much money would the manongs have left over after paying rent? What questions do you have about this situation? Is this a fair wage?

The students work in groups using a choral counting table to solve the problem. They record their group's solutions on chart paper.

***Summarize.***

One representative from each group shares the group's solution to the problem along with their questions and comments. The class discusses them.

***Act.***

With what they have learned, students can research the current hourly pay of migrant farmworkers and analyze the relationship between their pay and their living expenses. Students can write letters to the United Farm Workers Union to find out about current conditions for migrant farmworkers. Students could explore wages and salaries of jobs in their communities and/or jobs in which they are interested. Students analyze the relationship between wages and living expenses.

## Connecting Mathematics Lesson to Civic Reasoning and Discourse.

***Knowledge.***

This lesson involves students engaging in number and operations as well as analysis as they make sense of historic farmworker salaries and living expenses. We note that choral counting extends typical ways of counting out loud by charting the count and provides opportunities for students to consider what they notice about the number sequence, thus leveraging student ideas to build conceptual understanding of number (Franke et al., 2018). Additionally, such a routine promotes discourse and justification conducive to mathematical understanding.

***Epistemology.***

This task supports students in developing knowledge in a real-world context by making sense of quantities and how numbers grow. That context enables students to connect numbers to real-world situations in meaningful ways and to see that mathematics is relevant and useful for understanding the world around us.

***Dispositions.***

This task allows students to make sense of real-world data in a historical and contemporary context. They work in groups to share noticings and wonderings regarding the choral counting of hourly wages.

The students use what they learned from the small group activity to discuss what a fair wage is as a whole group. All of this contributes to students' sense of mathematical agency, in which they see themselves as mathematical sensemakers.

### *Ethics.*

By analyzing the wages of migrant farmworkers, students will have a better understanding of the impact of wages on living conditions and the ways in which the people who are responsible for producing our food are treated.

### *Developmental Needs.*

For children in upper elementary school, perceiving linear growth patterns can be a challenge. The ideas of proportionality underlying the ways in which hourly salaries add up are not obvious to many children at this point (e.g., that a 12-hour-day's salary is twice the salary earned after 6 hours). That is why students need time to explore the patterns in Table 2. And, learning to link their numerical observations to real-world conditions in meaningful ways offers additional challenges. Addressing these challenges prepares students to make meaningful use of school mathematics.

### *Language.*

In exploring issues like the ones in this lesson, students need to be able to use language to communicate in ways that make sense to them. The lesson does not ask students to use particular mathematical language; informal language and other forms of expression are supported and valued. The lesson asks students to share their ideas in words and gestures. It includes pictorial representations to support students' sensemaking. The teacher's responsibility is to support the students' engagement and help them use both informal and more formal mathematical language in their sensemaking. Students can compare and contrast different mathematical approaches actively (see Box A, MLR7: Compare and Connect).

## LESSON B: ENVIRONMENTAL JUSTICE

In the lesson "Examining Air Quality," students explore what air quality measurements are and how they relate to whether air is safe to breathe.[22] The students analyze air quality maps from different locations and decide whether it is safe to go outside.

### Lesson Outline.

#### *Launch.*

Students are shown a map of California with air quality readings. With the whole group, students discuss what they notice and wonder about the map, including what the numbers and colors indicate. Students discuss the meaning of the air quality ratings and why they are important.

#### *Explore.*

Students work in small groups. Each group receives a different air quality map to investigate. Students take turns sharing what they notice and wonder about the maps. Students return to the whole group to share small group exploration findings.

#### *Summarize.*

Students take a position and support it with evidence. Students look at an air quality map for their neighborhood and determine whether the air is good, moderate, or unhealthy. Students place stickers on an air quality number line to indicate where they think today's air quality would be located.

#### *Act.*

Students can talk about air quality with their parents, caregivers, and other adults in their lives. Students can pay attention to news reports about air quality.

### Connecting Mathematics Lesson to Civic Reasoning and Discourse.

#### *Knowledge.*

This lesson involves very young students engaging in number and operations, measurement, and data collection and analysis as they make sense of the information on the air quality maps. As the lesson notes, "Many children in kindergarten are learning numerals, number names, and rote counting by ones. They are making sense of quantities and how our number system is ordered, exploring ideas like how the more digits a whole number has, the bigger it is. Alongside this, they are reasoning about the meaning of amounts and are learning concepts to compare numbers such as 'more than' and 'fewer than'" (Koestler et al., 2023, p. 116).

#### *Epistemology.*

This lesson supports students in seeing that different ways to represent real-world phenomena (e.g., maps, numbers) can help them to understand the world around them. The classroom discussions help students see how exchanging ideas with others helps to understand things better.

#### *Dispositions.*

This task allows students to make sense of real-world data that impact their lives. They work in groups to share noticings and wonderings regarding the air quality maps. They discuss as a whole class the

[22] The complete lesson can be found in Koestler et al. (2023).

similarities and differences between the maps they explored. Lastly, they use what they learned from the small group and whole group discussions to make claims about the air quality of their own region. All of this helps students to see themselves as agentive learners.

***Ethics.***

By analyzing the data, students will discover disparities in the air quality of regions that are related to socioeconomic factors. Students will also be able to understand for themselves what the air quality ratings mean and how to behave based upon those ratings.

***Developmental Needs.***

From the earliest of days in school, students need to believe that mathematics makes sense, is related to the real world, and is doable. This task allows students to work together to make sense of the data in the maps, compare and contrast their findings with other students, and make conjectures related to the quality of air they breathe in their own communities.

***Language.***

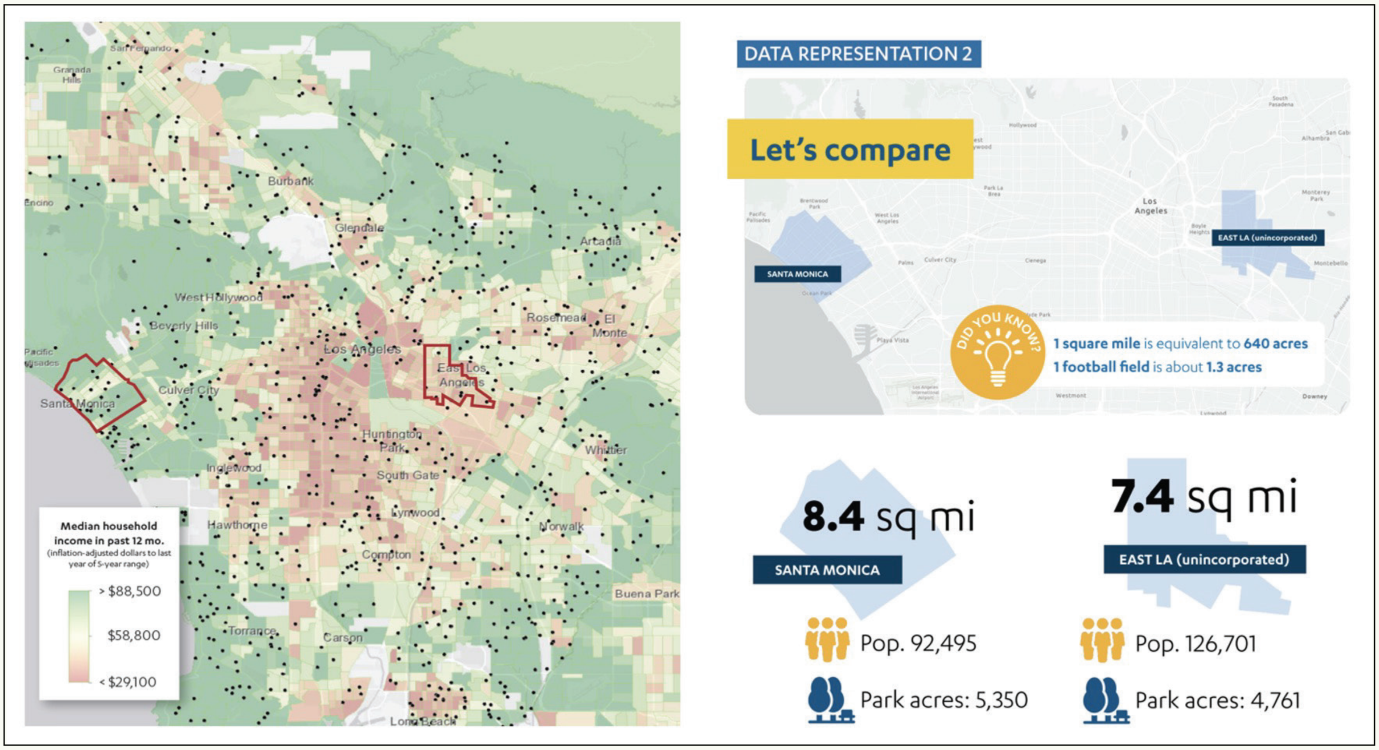
This task introduces students to a range of mathematical and other terms that require careful use of language. Students need to use and ground meanings for mathematical terms in actions, concrete objects, and/or representations. What does "more" mean, for example, and how do different meanings correspond to the written numerals or the colors used to represent air quality on the map? Sorting out such issues requires discussion (see Box A, MLR8: Discussion Supports) and attention to using vocabulary with meanings that are appropriate for very young students. the small group and whole group discussions to make claims about the air quality of their own region. All of this helps students to see themselves as agentive learners.

## LESSON C: SOCIAL JUSTICE

The lesson "Parks in Los Angeles" explores questions such as the following: Who has access to parks and who doesn't? Why do some communities have lots of park space while other nearby communities have almost none? Should there be more parks, and if so, where should they be located? What kind of services should parks offer?[23] The University of California, Los Angeles (UCLA) Data for Democracy in LA project[24] offers a template for discussions and a set of data representations to consider. These can be used to investigate inequality related to parks or as a template for issues that could be examined in readers' communities.

The intentions of the lesson are (a) to draw on data to support conversations around inequality and (b) to critically examine the data sources and determine trustworthiness. For example, the students can engage with a table that shows the median income and park acreage for 10 local communities. They are asked to consider where the data come from and whether they should trust them. Students can compare different data representations to see how they might highlight different aspects of the data. They can engage with a map that shows park locations and median income, and then explore a representation of two of those communities that vary in median income and the park acreage for each (see Figure 4). The data brief includes a wide range of both quantitative and qualitative data to support students in considering what can be learned from these different kinds of data.

***Figure 4:*** **Park Locations and Median Incomes**




*Note.* UCLA Data for Democracy Team, retrieved from https://ucla.app.box.com/v/dfdbrief1parks, with permission.

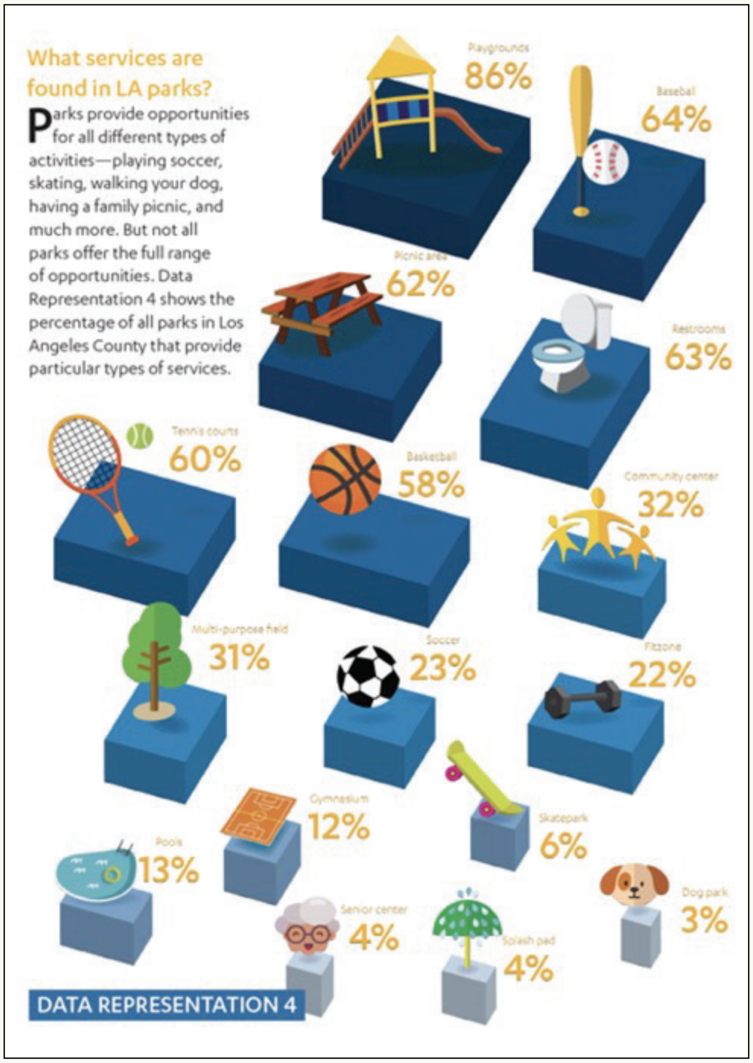
[23] The complete lesson can be found at https://centerx.gseis.ucla.edu/data-for-democracy/briefs/parks/
[24] See https://centerx.gseis.ucla.edu/data-for-democracy/

## Lesson Outline.

### *Launch.*

Students are asked about the parks in their neighborhood. How far do they have to walk to get there? What is available at their parks? Do they have soccer fields or bathrooms? Do the students ever drive far to get to a park they really like? What does that park have that they like? The intention here is conversation—learning what students are interested in related to parks rather than collecting specific data from the students.

*Figure 5:* **Services Available in L.A. Parks**




*Note.* UCLA Data for Democracy Team, retrieved from https://ucla.app.box.com/v/dfdbrief1parks, with permission.

### *Explore.*

The teacher shares the representation showing the services available at parks in Los Angeles (or replaces it with one from the students' community) (see Figure 5). Students are asked what they notice and what they wonder about. What surprises them? Why might that be?

The class pursues comparisons of what services are most available and least available. They might consider questions like, "If only 63% of parks have bathrooms or 23% have soccer fields, where do you think the parks are located that have bathrooms or soccer fields? Are there certain features that you think are essential for all parks?"

Students can also consider questions such as the following: Which park service is found almost twice as often? If there are 1,587 parks represented in this data set, how many more parks have baseball fields than bathrooms? For a mathematical challenge that involves proportional reasoning, they can consider this issue: If you want to build 10 new parks with the same representation of services that now exists in LA parks, what services would the parks have? What if you were building 100 more parks?

### *Summarize.*

Students are asked to make sense of these data in relation to their own parks. What services do they think would be most important for their community? Why? Do they think other students at their school would agree with them, or do they think some students would want other things at their parks?

### *Act.*

Opportunities exist to learn more about the parks in the community. Students can choose a question they want to answer: Which parks in our community have bathrooms? Where are they? Are they in communities where income is higher? Are they in communities that have more park acres? How can we find answers to those questions? Where would we look, or who could we ask?

## Connecting Mathematics Lesson to Civic Reasoning and Discourse.

### *Knowledge.*

This lesson allows teachers to choose their mathematical focus. It asks students to make sense of a representation and extend the data in ways that ask for them to generalize. The data conversation can be used to compare percentages, to consider percentages in relation to quantity, and to reason proportionally.

### *Epistemology.*

These data ask students to consider how space in communities is designated and who has access to what kind of space, starting with spaces that students most likely would have experienced. The lesson asks students to wonder why that may be, what space would be important for everyone, and who makes those decisions. Furthermore, it is important to situate parks historically and recognize that parks may sit on Indigenous lands. The lesson helps students to make connections between school mathematics and “real-world” issues.

### *Dispositions.*

The goal of this lesson is to provide students the opportunity to see how to use data and mathematics to make sense of their world and take action. The intention is to engage students in inquiry-oriented, student-driven learning and conversations about what they think is important. Such learning builds on students’ innate curiosity about issues in their communities and fosters dispositions such as investigating evidence and weighing multiple points of view.

### *Ethics.*

This lesson provides ample opportunities for students to engage with different types of data, determine the trustworthiness of information and data sources, and construct an objective narrative of a social issue. The lesson also encourages students to investigate disparities in park access.

### *Developmental Needs.*

Elementary students are creative problem-solvers. This lesson takes advantage of that and enables students to draw on their experiences and understandings to make sense of data, have conversations about inequality tied to those experiences, and wonder about different communities. It asks students to consider that not all students experience the world in the same way or agree with each other, and it helps them to refine their thinking in conversation with others.

### *Language.*

In exploring issues like the ones in this lesson, students need to be able to use language in ways that make sense to them. It does not ask students to use particular mathematical language; informal language and other forms of expression are supported and valued. The lesson asks students to s hare their ideas in words and gestures (see Box A, MLR8: Discussion Supports). It uses a pictorial representation to support their sensemaking. The teacher’s responsibility is to support the students’ engagement and help them use both informal and more formal mathematical language.

## Issues and Challenges Particular to Middle School Mathematics

Many middle school mathematics students begin to lose interest in the discipline and will, when possible, make the choice to leave mathematics as soon as they can. This trend continues into the secondary school grades. The challenge is to maintain student interest. Addressing relevant " real-world" issues in meaningful ways is one way to do so, especially if the students can draw upon their real-world knowledge to help make sense of the issues they are studying.

There are many considerations from the student's point of view. Students are their "whole selves" when they enter the classroom—they don't leave their moral selves, their social selves, or their other personal commitments at the door. In developmental terms, adolescents are developing and refining their sense of moral reasoning. Peer opinions matter a great deal to them; issues of attitude and confidence arise, especially given gender and racial stereotypes. Hormones begin to kick in, along with myriad insecurities and issues of identity formation. As Haddock et al. (2018) note, middle school students are simultaneously coping with physical, social, and emotional changes. Their identities are in flux and are often fragile.

The school context also changes significantly. In elementary school, most students stay with one or two homeroom cohorts for much of the day. In middle school, they often spend time in disciplinary silos—math, science, English, social studies, etc. There are also age-specific issues of language. By the time students enter middle school, they have acquired their first language (Valdés, 2001). This can serve as a resource, whether or not that first language is English. In general, everyday language is a resource for problem-solving, sensemaking, exploring new topics, sharing beginning explanations with peers, and sorting out misunderstandings. Learning to communicate about new mathematics (e.g., sharing one's reasoning on topics such as algebra); constructing, revising, and critiquing new kinds of arguments; and learning to use more formal registers are some of the new goals for math instruction in middle school—goals that need to be negotiated carefully.

While elementary school teacher preparation has a significant focus on the "whole child" and developmental issues, and secondary school preparation has a strong disciplinary focus, effective teaching in middle school—actually, all grades—calls for significant attention to both. Middle school mathematics teachers can benefit from additional professional development opportunities in addressing potentially controversial social issues, both from the curricular side (complex, value-laden issues of mathematical modeling) and the interpersonal side (where managing the emotions that can emerge in fraught discussions can be a significant challenge). Curricular pressures begin to accelerate, with more and more content coverage each grade and high-stakes testing. These challenges must be faced if room is to be made for the kinds of tasks that support civic discourse that are discussed in this report.

## LESSON A: FAIRNESS

In the lesson "Gender Gap and Pay," students discuss differences between the salaries of those who identify as males and females by examining the percent change of salaries across various careers.[25] They then do a more in-depth study of a particular career of their choice in order to investigate the variables potentially responsible for these differences (variables can include experience, postsecondary education, age, etc.). Students use a range of mathematical representations to explore gender pay gaps and discuss issues related to them.

### Lesson Outline.

#### *Launch.*

Students work in small groups to determine the median salaries of males and females in a range of professions. They explore the use of median as opposed to other measures (e.g., mean) to represent a body of data fairly, and learn to compute salary differences in terms of percentages. They discuss possible causes of gender pay gaps.

#### Explore.

Students select one job category for intensive exploration. They then examine scatterplots of male and female salaries in their chosen profession over time and determine best fit lines to represent salary trends. Students make posters to demonstrate what they have found.

#### *Summarize.*

The class engages in an organized discussion of the posters, after which it considers the efficacy of the mathematical representations and discusses various parameters used in those representations.

#### *Act.*

Students are encouraged to discuss possible career choices and the impact of their explorations on those choices. Extension activities include researching further job categories, reflecting on the career of Ruth Bader Ginsburg, and doing research on equal pay legislation.

### Connecting Mathematics Lesson to Civic Reasoning and Discourse.

#### *Epistemology.*

Classroom discussions open up explorations of the factors that influence how salaries are set as well as social factors that may determine salaries by gender. This leads to the possibility of developing understandings regarding the social construction of inequities.

#### *Dispositions.*

The activity provides students with tools for inquiring into possible careers of interest and for understanding social issues related to them. This has the potential to foster an increased sense of agency.

#### *Ethics.*

Wage gaps and their causes are significant ethical issues. This lesson provides a solid data-based way to enter into discussions and problematize the current state of affairs.

[25] The complete lesson can be found in Conway et al. (2022).

### *Developmental Needs.*

Middle school students are first learning to grapple with multivariate phenomena and the idea that there may be a range of causes for single-variable outcomes. Coming to grips with the fact that phenomena are not simply deterministic, and that there are meaningful trends, is a significant challenge.

### *Language.*

Before embarking on finding solutions for this lesson, students first need to have the time and the opportunity to understand the problem well enough to dig into it (see Box A, MLR6: Three Reads). This task provides opportunities to compare and contrast different mathematical approaches (see Box A, MLR7: Compare and Connect). Instruction can also support students in actively revising their ideas, explanations, or arguments (see Box A, MLR1: Stronger and Clearer Each Time).

## LESSON B: ENVIRONMENTAL JUSTICE

The lesson "The Mathematics of Toxic Air Emissions" explores the impacts of toxic air emissions on an economically disadvantaged neighborhood in the midwestern United States.[26] In this lesson, students explore different representations of numerical data and what information can be gleaned from those representations. Specifically, students are expected to use the information from a table containing toxic air emissions and average income by zip code, choose an appropriate graph to represent the data, graph the data, and make observations and conclusions based on the data.

### Lesson Outline.

#### *Launch.*

Students read a worksheet with data concerning toxic air emissions in Cincinnati, Ohio. They discuss issues raised by the information in the worksheet and then consider potential uses of mathematics for representing or misrepresenting important information.

#### *Explore.*

The students discuss possible relationships between categories of data in the table (zip codes, toxic air emissions, and average household income). Students create a graph using "average household income" and "toxic air emissions," and then explore particulars: What is the average income in the area with the largest amount of toxic air emissions and with the smallest amount? They learn about the impact of the emissions on people's lives and discuss the correlations in the data.

#### *Summarize.*

In the summary discussion, students review what they learned from their graphs and whether they discovered inequitable patterns. They are led to explore data for their own neighborhoods and discuss the implications.

#### *Act.*

Students are encouraged to write letters to their local and federal environmental protection agencies about areas that have bad air quality, and to raise awareness of the issue to others in the school and in the community through local and social campaigns about clean air.

### Connecting Mathematics Lesson to Civic Reasoning and Discourse.

#### *Knowledge.*

The mathematics in this lesson involves students translating from one visual representation of numerical data to an alternative representation of those data. Specifically, students are expected to use the information from a table containing toxic air emissions and average income by zip code, choose an appropriate graph to represent the data, graph the data, and make observations and conclusions based on the data. More broadly, this kind of information (that the environmental quality of life may be a function of income) may be new to students.

#### *Epistemology.*

These data provide nuances to simple assumptions about life. Neighborhoods where people live have different characteristics. Realizing that where people live can have a direct impact on their health because of environmental factors may be new, and such learning may uncover different ways of thinking about social justice.

[26] The complete lesson can be found in Conway et al. (2022).

***Dispositions.***

The activities described in "Act" may help students to see that it is possible to act on information they learn, which gives them a sense of agency. The lesson as a whole may reinforce the idea of mathematics as a discipline that helps to make sense of and act on aspects of one's personal life.

***Ethics.***

The information that basic health risks may correlate with income—information that needs to be known and acted upon—may be a revelation. The two articles in Box E, "How Air Pollution Across America Reflects Racist Policy From the 1930s"[27] and "Redlining means 45 million Americans are breathing dirtier air, 50 years after it ended,"[28] indicate that this is an arena ripe for exploration.

***Developmental Needs.***

Middle school students are moving from simple right-or-wrong views to more complex understandings of issues and the tensions involved when trying to resolve them. They need to learn and believe that their understandings and actions can have societal impact, and also to find a way to put such understandings in perspective.

***Language.***

In exploring issues like the ones in this lesson, students need to begin by using language to communicate in ways that make sense to them; informal language and other forms of expression should be supported and valued. The teacher's first responsibility is to support the students' engagement and help them use both informal and more formal mathematical language in their sensemaking (see Box A, MLR2: Collect and Display). Beyond those initial sensemaking discussions, instruction can support students in learning to revise their arguments (see Box A, MLR3: Clarify, Critique, Correct).

[27] https://www.nytimes.com/2022/03/09/climate/redlining-racism-air-pollution.html
[28] https://www.washingtonpost.com/climate-environment/2022/03/09/redlining-pollution-environmental-justice/
[29] The complete lesson can be found in Gutstein & Peterson (2013).

## LESSON C: SOCIAL JUSTICE

### Lesson Outline.

#### *Launch.*

Students are asked if they have any experience with being racially profiled or if they know anyone who has been racially profiled. Students share their experiences and/or knowledge of racial profiling.[29]

#### *Explore. Part 1:* Finding California's Racial Breakdown.

Each group of students is given a small bag with colored cubes to represent the racial breakdown of California. The bags contain 19 tan cubes (for Whites, 38%), 19 red cubes (for Latinx, 38%), seven blue cubes (for Asians, 14%), three black cubes (for African Americans, 6%), and two green cubes (for others, 4%) to approximate California's racial proportions. Teachers should not tell the students the total number of cubes or how many of each color there are.

Students pick one cube from the bag without looking. They record its color on the chart (tally marks work well) and replace the cube. Repeat this 100 times. After every 10 picks, the students record totals on a chart. Each line of the chart is the cumulative total of picks. Students are told that they are conducting an experiment (picking/replacing 100 times), collecting data (recording each pick), and analyzing data (determining for their simulation how many cubes of each color are in the bag, the total number of cubes, and the California racial/ethnic percentages).

It is important for the students to record the fraction and the percentage of each race/ethnicity for every 10 picks on the chart.

Each group of students is asked to provide explanations for the following questions:

1. What do you think is in the bag? Why do you think this?
2. What happened as you picked more times, and what do you think would happen if you picked 1,000 times?

#### *Explore, Part 2:* Investigating Driving While Black and Brown.

Students are provided the following data from the Los Angeles Times analyzing traffic stop data from 2012 to 2017: Of 378,000 drivers stopped by the California Highway Patrol (CHP), 151,200 were Latinx.

During the same period of time, the Domestic Highway Enforcement Team (DHET), a team of Los Angeles County sheriff's deputies who cruise the freeway stopping motorists in search of cars carrying drugs, pulled over about 13,333 people. Of those stops, 9,200 motorists were Latinx.

Students are asked to use what they learned in the first experiment to set up their own simulation of this situation using cubes. They are instructed to pick and replace a cube 100 times, record the data, and calculate the results of simulating 100 stops.

Students address the following questions in groups:

1. What percentage of the motorists pulled over by the CHP were Latinx?
2. What percentage of the motorists pulled over by the DHET were Latinx?
3. How did you set up the simulation for the investigation (how many "Latinx" cubes and how many total)? Why did you choose those numbers?

4. In your simulation, how many "Latinx" cubes were picked out of 100 picks, and what percentage is that?
5. Do the results of the simulation investigation support the claim of racial profiling? Why or why not?

***Summarize.***

Students are asked the following questions:

1. What did you learn from this activity?
2. How did the mathematics help you do this?
3. Do you think racial profiling is a problem, and if so, what do you think should be done about it?
4. What questions does this project raise in your mind?

The lesson ends with a whole class discussion of these issues.

***Act.***

Students use responses from Question 3 in "Summarize" to develop a plan of action around racial profiling. For example, students can reach out to community organizations that deal with police accountability. Students also can contact their local law enforcement organization to determine its policies with respect to racial profiling. Students can contact their elected officials to advocate for laws against racial profiling.

## Connecting Mathematics Lesson to Civic Reasoning and Discourse.

***Knowledge.***

The content knowledge covered in this lesson is probability. Issues of experimental versus theoretical probability are addressed through this lesson. As described above, these understandings are used to assess the likelihood that police stops are random.

***Epistemology.***

Students are given a scenario related to police stops by race and are required to create their own probability simulation to determine if the scenario represents a case of racial profiling. This helps students understand that mathematics can be used to assess the likelihood of certain "real-world" events and to uncover biases in an objective and scientific way.

***Dispositions.***

Students are challenged to assess information about police traffic enforcement stops from multiple perspectives. While many students may have opinions about racial profiling based upon their personal experiences or what they have seen or read in the media, it is important that students base their decision about whether or not the scenario represents a case of racial profiling on the mathematics.

***Ethics.***

Students may come to different conclusions based upon the results of the probability simulation. It is important to have students base their conclusions on the mathematics involved. At issue here are underlying ethical commitments to the unbiased application of the law and the neutral application of mathematical methods to determine what can be judged mathematically as biased or unbiased.

### *Developmental Needs.*

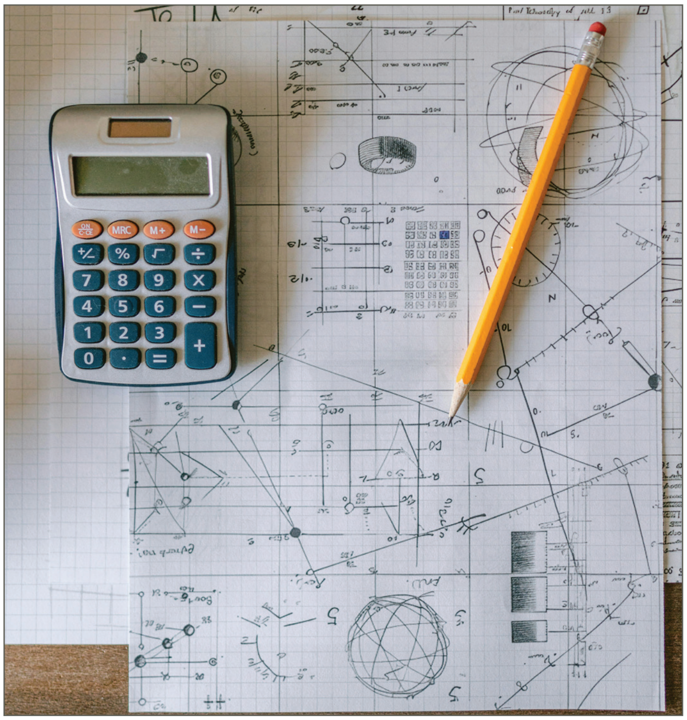
This kind of unit can help develop a sense of agency, where students see that they can uncover injustice in compelling ways. However, the issues dealt with in this unit are challenging personally and mathematically. Dealing with issues of bias is extremely difficult in terms of identity, especially during times of identity formation. Seeking somewhat objective tools to determine bias may not seem like a "natural" thing to do.

### *Language.*

In exploring issues like the ones in this lesson, students need to begin by using language to communicate in ways that make sense to them; informal language and other forms of expression need to be supported and valued. The teacher's first responsibility is to support the students' engagement and help them use both informal and more formal mathematical language in their sensemaking (see Box A, MLR5: Co-Craft Questions, for another way to support students' active sensemaking and problem-solving while attending to language). This can be done orally or in writing to support revising. Beyond those initial sensemaking discussions, instruction can support students in learning to revise their arguments (see Box A, MLR3: Clarify, Critique, Correct).

## Issues and Challenges Particular to Secondary School Mathematics

The mathematics students encounter in secondary school becomes increasingly abstract and technically difficult. This presents an opportunity and a challenge. The opportunity is that secondary school mathematics is remarkably powerful, with tremendous opportunities for sensemaking and meaningful applications to important real-world problems. The challenge is that students do not often experience mathematics this way. As discussed above in "Middle School," many students in both middle and secondary school lose interest in the discipline and make the choice to leave mathematics as soon as they can. Addressing relevant "real-world" issues in meaningful ways is one way to maintain student interest. And, of course, adolescents rebel. The realities of the world (e.g., job possibilities, racism, and issues of social justice) really begin to hit home for students. On the one hand, this means that the careful consideration of such issues opens up possibilities for meaningful discussions in mathematics classrooms. On the other hand, such issues are delicate and must be addressed with appropriate care.




There are age-specific issues of language. By the time most students enter secondary school, they have acquired their first language. This can serve as a resource, whether or not that first language is English. At the same time, communication in mathematics classrooms has its own demands, and those need to be negotiated carefully.

The secondary school context differs substantially from that of middle school. Tracking in mathematics classes becomes a more significant issue at the secondary level, with implications for equity and for student identities. Most secondary mathematics teachers teach mathematics only. Their teacher preparation training or professional development may not provide sufficient opportunities to address social issues, both from the curricular side (complex, value-laden issues of modeling) and the interpersonal side (where managing the emotions that can emerge in discussions can be a significant challenge). In addition, greater curricular pressures exist—an increasingly fast-paced curriculum, demands for more and more rapid content coverage, and high-stakes testing that tends to narrow the focus of instruction to what is tested. All of these challenges must be faced to make room for the kinds of tasks that support civic discourse that are discussed in this report.

## LESSON A: FAIRNESS

In the lesson "Do Postal Codes Predict Test Scores?" students explore edgap.org, which provides graphical representations of average high school SAT and ACT scores, median household income, and other socioeconomic indicators in a range of districts across the United States.[30] The students then choose variables of interest to them and explore correlations between those variables and SAT and ACT scores.

### Lesson Outline.

#### *Launch.*

Students work in groups with an internet-connected device to explore edgap.org. They take 10 minutes to write down noticings and wonderings. Groups share their noticings and wonderings. To assure basic understandings, students are probed as to what the different numbers and colors on the map represent.

#### *Explore.*

Students work in groups to decide what data they want to analyze from the four areas provided by edgap.org: median household income, percent unemployment rate, percent of children in a two-parent household, and percent of college-educated parents.

The students then use edgap.org to create a table of 20–30 schools of interest, recording the ACT/SAT score and their variable of choice. They use the data from the table and technology to create a scatterplot and model for their data, after which they determine the correlation coefficient r. The data points on their scatterplots are labeled by school name.

#### *Summarize.*

Students do a gallery walk to share findings. Students notice similarities and differences in their graphs, models, and correlation coefficients. Students discuss any economic and/or racial trends and disparities in the data.

#### *Act.*

Students brainstorm who they might write a letter to in response to their data analysis. They work together in groups to compose letters to the people on the brainstorm list.

### Connecting Mathematics Lesson to Civic Reasoning and Discourse.

#### *Knowledge.*

This lesson taps into a range of knowledge related to scatterplots as a mechanism for analyzing the association between two variables. The visual representations of different correlations allow students to see how strong the correlations are (the visual counterpart to correlation coefficients).

#### *Epistemology.*

This task allows learners to become aware of biases in standardized testing. This is an important understanding—numbers like test scores are not absolute measures of someone's knowledge or potential, but several factors outside a person's control may affect test scores. Because the overall task is multivariate, it illustrates the complexity of working with real-world issues rather than the standard "there is one right answer to every math problem" impression that students may develop from the curriculum. Students can see that one variable can have some impact on another but only in the aggregate and not in a uniform way.

[30] The complete lesson can be found in Berry et al. (2020).

### *Dispositions.*

This task reinforces several dispositions essential for the thoughtful modeling of real-world phenomena. For example, students have the opportunities to weigh competing evidence: which test (if either) reflects proficiency? What kind of proficiency? Which variables matter, in which ways? Students also play with variation as a way of approaching complex phenomena. The learners create a data table of 20–30 data points using their variables of choice, graph their data, and use technology to determine the correlation coefficient r. Different choices of data points result in different values, and students will need to sort out these differences. Similarly, they will need to sort out the impact of the four different variables they compare collectively—median household income, percent unemployment rate, percent of children in two-parent households, and percent of college-educated parents. The learners do a gallery walk to share findings and look for patterns in class graphs. In doing so they need to listen to each other and sort out commonalities and differences.

### *Ethics.*

The data are likely to challenge some fundamental assumptions about testing. Standardized tests are supposed to be objective measures of student ability, but the data seem to suggest that students from higher-income backgrounds score higher on standardized tests than students from lower-income backgrounds. That finding can raise a range of ethical issues. For example, if the tests are unfair, should they be thrown out? Is there a way of compensating for test bias? If the compensation is statistical, will it be fair to individuals? What are more equitable ways to determine what students know?

### *Developmental Needs.*

Students learn to collect and organize data that are important to them. They grapple with complexity and make sense of data. In doing so, they may have opportunities to develop self-efficacy. These data may be relevant to the students personally (or their friends), if they see themselves reflected in the data pool. The lesson may raise questions for them about their own biases and perceptions about testing, and about society's biases. And, it prepares them to confront subtle issues of bias in society and to discuss such issues thoughtfully.

### *Language.*

Before embarking on finding solutions for this lesson, students first need to have the time and the opportunity to understand the problem well enough to dig into it (see Box A, MLR6: Three Reads). This task provides opportunities to compare and contrast different mathematical approaches (see Box A, MLR7: Compare and Connect). The teacher can ask students to generate mathematical questions or problems for the situation, using this lesson to create space for students to produce the language of math questions themselves and develop meta-awareness of the language used in mathematical questions and problems (see Box A, MLR5: Co-Craft Questions).

## LESSON B: ENVIRONMENTAL JUSTICE

In the lesson "Climate Change in Alaska," students examine 100 years of real data from Alaska that chronicle when seasonal ice cracked enough to allow for bridge or other construction.[31] They use scatterplots and various algebraic mechanisms to model the timing of ice breaks and make predictions.

### Lesson Outline.

#### *Launch.*

Students are introduced to the Nenana Ice Classic, including a description of the origins of making predictions on when the river's ice block will break up, and they are given a brochure with more details. They then visit a website that gives information related to this year's contest. The students discuss and explore patterns in the data. They are then charged with predicting the day and time when the ice in the Tanana River at Nenana will break. They work in groups to make predictions.

#### *Explore.*

Students use the data provided from the brochure and website to create a graphical display and model that can be used to predict when the ice will break the following year. They use technology to plot their data and fit an appropriate model to their scatterplot. Students discuss trends that they see in the data and their models. They make conjectures regarding why the ice seems to be breaking earlier as time progresses.

#### *Summarize.*

Student groups share their scatterplots and models. They interpret the parameters of their models in the context of climate change in Alaska.

#### *Act.*

Students brainstorm actions they can take to deal with climate change (e.g., recycling, using reusable products, using public transportation). Students choose which action they want to take and complete that action as a final project.

[31] The complete lesson can be found in Berry et al. (2020). Resources for the lesson can be found at https://resources.corwin.com/tmsj-highschool/student-resources/chapter-6/lesson-62-climate-change-in-alaska

## Connecting Mathematics Lesson to Civic Reasoning and Discourse.

### *Knowledge.*

This lesson involves the use of equations and inequalities to make predictions. Students analyze the relationship between two variables using statistical procedures, including determining the quality of a linear fit to data, generating least-squares regression lines, finding the correlation coefficient, and exploring the difference between correlation and causation. In short, students develop knowledge about and skills in assessing statistical procedures.

### *Epistemology.*

The task helps students understand that prediction is not an exact science but that models enable one to grasp complex phenomena and help shape our construction of knowledge. Students learn to sort through complex phenomena to find variables that seem to matter, evaluate criteria, and test whether they actually do matter (and to what degree) in empirical fashion. This is a form of dealing rigorously—but not exactly, as in classical mathematics—with issues of modeling and prediction.

### *Dispositions.*

This task reinforces several dispositions essential for civic reasoning and discourse. Students deal with multivariate phenomena, which provide an opportunity to view multiple perspectives. They work in groups to make predictions, a process that involves collective reasoning, cooperation, and a form of give-and-take that is useful for civic discourse. Specifically, groups share initial thoughts after thinking about the problem for a few minutes but before they have come up with a prediction. After each group has shared, the groups continue to work on their predictions using what they heard during the group sharing. This reinforces that collaboration is an effective mechanism for problem-solving.

### *Ethics.*

By analyzing the data, students explore the reasons why the river ice is breaking earlier each year and the implications if the ice continues to break earlier each year. This raises a series of issues regarding the impact of human behavior on the climate and what is right and just. The short HBO video "Climate Change is Making This Alaskan Town Fall into the Ocean" underscores the ethical issues involved.[32]

### *Developmental Needs.*

The unit helps students develop a sense of communal responsibility for actions, with the understanding that individual actions contribute in a cumulative way both to causing problems and potentially addressing them. They have opportunities to develop self-efficacy both in choosing and analyzing data, and in contemplating the impact of actions taken to address the issues of climate change. They use their knowledge to propose mechanisms for making progress, which is a contributor to personal identity and empowerment, as opposed to simply arriving at "right answers" on math problems.

### *Language.*

Before embarking on finding solutions for this lesson, students first need to have the time and the opportunity to understand the problem well enough to dig into it (e.g., see Box A, MLR6: Three Reads). Students can share their oral contributions while the teacher listens for and transcribes the student output using written words, diagrams, and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. The teacher can provide feedback for students to increase accessibility and also support meta-awareness of language (see Box A, MLR2: Collect and Display).

[32] https://www.unifiedfilmmakers.com/medialibrary/climate-change-is-making-this-alaskan-town-fall-into-the-ocean-hbo/

## LESSON C: SOCIAL JUSTICE

In the lesson "How Do Poor People Bank?" students explore how certain kinds of profit-making financial institutions are prevalent in low-income communities and what mechanisms such institutions use to take advantage of people's difficult financial situations.[33] Students investigate the impact of compound interest on loans.

## Lesson Outline.

### *Launch.*

Students share their prior knowledge about interest. Students give examples of when people pay interest and when interest is paid to them. The term "interest" is defined, and an example is given. The concepts of compound interest and annual percentage rate are introduced. Students are given the compound interest formula $A = P(1 + r/n)^{nt}$. They are given an example that requires them to use the compound interest formula.

Next, students are introduced to ways that compound interest can work against them. Examples include check cashing establishments, payday loans, rollover loans, rapid tax refunds, money transfer agents, and credit cards.

### *Explore.*

Students are given a set of six problem scenarios involving compound interest to solve in groups.

### *Summarize.*

Students share solutions to the problems they solved. They respond to the following questions: What is the problem with this system? How is this bad for people? How does this keep people poor?

### *Act.*

Students write letters to elected officials arguing for the need for regular banking institutions in all communities.

[33] The complete lesson can be found at https://www.radicalmath.org/news-1/banking-options-in-low-income-communities

## Connecting Mathematics Lesson to Civic Reasoning and Discourse.

### *Knowledge.*

This lesson involves percent, interest and compound interest, exponents, and exponential growth. This is both core mathematical content and essential mathematics for financial literacy.

### *Epistemology.*

Working through the issues in this lesson requires evaluating evidence and examining assumptions underlying the evidence. That is, while the use of mathematics is "neutral" in that it "simply" computes interest, issues of what interest rates are available to which parts of the population are not neutral and contribute to injustice.

### *Dispositions.*

The unit indicates that certain questions are fair, if not essential, to ask: How do poor people bank? Why do poor people rely so heavily on expensive financial systems? What are the financial costs of using these methods to bank?

It supports weighing competing evidence: Through the concept of compound interest, students explore the interest rates at a variety of financial institutions, specifically banks versus check cashing/payday loan businesses.

Finally, students examine multiple points of view: They share their results from problem-solving and their conclusions about the access to banking services in different communities.

Students listen to others: They share their experiences with banks versus check cashing/payday loan businesses in their communities.

All of this contributes to the understanding that what is learned in math class can be used to understand injustices as they play out in society.

### *Ethics.*

This task helps students identify the social injustices of banking and lending in low-income communities of color and consider ways just decisions can be made. Students confront issues such as the fact that many banks refuse to put branches in low-income communities, causing these communities to rely on check cashing businesses and payday loan establishments that charge much higher interest rates than banks. This causes disparities in wealth in different communities.

### *Developmental Needs.*

The lesson contributes to students' understanding of the financial conditions in their communities, potentially enhancing their sense of civic identity. It also contributes to self-efficacy: Students are able to analyze the interest rates charged by financial institutions, which is a useful, if not necessary, life skill. More broadly, as a matter of their emerging sense of citizenship, students gain the knowledge to determine whether financial institutions are serving the needs of the community.

### *Language.*

Before embarking on finding solutions for this lesson, students first need to understand the problem. To this end, students can share their contributions while the teacher listens for and transcribes the student output using written words, diagrams, and pictures. The output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. The teacher can provide feedback for students to increase accessibility. The teacher also can ask students to generate mathematical questions or problems for the situation, using this lesson to create space for students to produce the language of math questions themselves and develop meta-awareness of the language used in mathematical questions and problems (see Box A for the full set of MLRs, several of which could be used profitably in this lesson, such as MLR6: Three Reads; MLR2: Collect and Display; and MLR5: Co-Craft Questions).

## PART 1: CONTEXT AND PROFESSIONAL DEVELOPMENT

The large-scale implementation of the ideas in this report will require two types of systemic support. The first relates to the contexts within which teachers work. First, teachers will require "permission" to engage with the kinds of materials discussed here. In some political jurisdictions, the topics emphasized in our examples might simply be off-limits. Even if the political climate is supportive, teaching these lessons takes time, which is a scarce resource. While important mathematics is in the lessons and is typically assessed, connections to "real-world" issues typically are not assessed. In consequence, many of the understandings that students develop, no matter how valuable, will not be credited in official reports of student learning. Second, teachers will require supportive materials and professional development. Also, creating a safe space for discussions of potentially contentious issues and helping students explain their thinking and listen to others respectfully as they do so call for teachers' skills and understandings that go beyond those found in the typical mathematics classroom.

In the introductory sections of this report, we identified some of the challenges that the field as a whole needs to address in order for teachers to be willing to move in the directions highlighted in this report. Specifically, mathematics education–focused organizations as well as funding agencies will need to address questions such as the following:

- What larger institutional support is available for moving in these directions?
- How can such activities be embedded in curricula?
- What kinds of curricula materials support use in classrooms with minimum support?
- How can the range of epistemological and dispositional issues highlighted in this report be assessed?
- How can teachers justify the use of these curricular materials on their own?

Beyond that, a question remains about providing materials and professional development to support teachers. Minimal professional development consists of working through some examples and showing how the patterns of engagement used in these materials (consistent with the principles and practices enunciated earlier in this report) can be applied to all math lessons. But the real payoff occurs when the contexts within which teachers work allow the use of these lessons.

Mechanisms need to be developed that point to ways in which teachers, professional learning communities, districts, and coaches can take ownership of these ideas and develop them so that they are not imposed from "above." For example, this is the type of work that could benefit from school-based, teacher-driven, collaborative, and student-focused lesson study as teachers experiment with implementation of these civic engagement tasks.

## PART 2: FUNDING

The kinds of mathematically rich activities related to civic reasoning and discourse in mathematics classrooms are unlikely to occur at scale unless some form of institutional backing for this kind of effort emerges.There are various possibilities:

- Practitioner organizations might publicize and/or support professional development workshops using the examples in these reports.
- The National Science Foundation or other funding agencies might be amenable to similar workshops, especially if endorsed by practitioner organizations.
- It might be possible (with funding) to build video-supported packages that would allow professional learning communities (including school districts) to take on such professional development themselves.
- It might be possible to conduct research and development on materials intended to support these aims.[34]

To find support for projects in these directions, practitioner organizations could generate ideas for professional development and various foundations could provide support for pilot projects.

Lastly, state-based organizations could provide opportunities for dissemination. For example, the statewide NCTM affiliate California Mathematics Council could offer professional learning opportunities around civic education and math in its various professional development institutes and workshops.

### *Case Study: Formative Assessment Lessons*

The Formative Assessment Lessons (FALs), built by the Mathematics Assessment Project, serve as an example of how substantial funding support can support classroom teaching and yield significant student learning gains. The FALs include 100 two- to three-day lessons.[a] The lessons contain 20-page lesson plans/scripts that help teachers understand the central issues related to student thinking. The materials made a significant difference in student learning. Teachers who were provided some professional development and then taught four to six lessons (approximately equivalent to 10–12 days of instruction) saw content gains of 4.6 months on average (Herman et al., 2014). But, those lessons could not have been developed without significant grant support.

[a] Available at https://www.map.mathshell.org/lessons.php

[34] We note that specific materials related to the TRU Framework are available on the TRU website (https://truframework.org/) and that two new books with tools for professional development (Schoenfeld et al., 2023a,b) are now available.

This appendix provides resources for supporting language in math classrooms.

Recommendations from research for attending to language include the following:

- Support sensemaking
- Optimize student output (both oral and written)
- Cultivate conversation, and
- Maximize linguistic and cognitive meta-awareness (Zwiers et al., 2017).

The following are some important pitfalls to avoid (Moschkovich, 2014, 2016, 2018):

- Do not simplify language, amplify it (Walqui, 2006; Walqui & Bunch, 2019)
- Do not focus only on words or vocabulary, but instead focus on math practices
- Involve and support multiple language functions
- Do not focus only on formal language, but instead build on students' informal ways of communicating and carefully consider when and how to support more formal language.

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### *Resource #1:* THE ENGLISH LANGUAGE PROFICIENCY DEVELOPMENT (ELPD) FRAMEWORK

As support for the use of best practices in science and mathematics classes, the Framework for English Language Proficiency Development Standards Corresponding to the Common Core State Standards and the Next Generation Science Standards (Council of Chief State School Officers, 2012), commonly known as the ELPD Framework, describes the multiple language functions associated with the Common Core State Standards math practices. Language functions, receptive or productive, are what students do with language to accomplish a math task. For connections between language functions and mathematical practices, see the math tables in the ELPD framework (pp. 20–25).

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### *Resource #2:* MATERIALS FROM THE UNDERSTANDING LANGUAGE PROJECT

Supporting English Language Learners (ELLs) in Mathematics: The goal of these materials is to illustrate how Common Core–aligned math tasks can be used to support math instruction and language development for ELLs at three grade spans (elementary, middle, and high school). Tasks were used or adapted from two publicly accessible curriculum projects, Inside Mathematics and Mathematics Assessment Project. To learn more, see the annotated lessons below and read the Guidelines for Math Instructional Materials Development: https://ul.stanford.edu/resource/supporting-ells-mathematics

"Language of Math" Task Templates: These are language-focused activities that can be used by teachers to design and write their own language-focused activities. These "Language of Math" templates were designed to support students in learning to read and understand word problems, communicate about mathematics, build disciplinary and academic vocabulary, and develop practices in mathematics. The templates can be found at https://ul.stanford.edu/sites/default/files/resource/2021-02/4.Language%20of%20Math%20Task%20Templates%2010-4-13.pdf

Another version of MLR6: Three Reads titled "Reading and Understanding a Math Problem," a task for supporting academic literacy with word problems, is on pp. 37–40 in the document "Language of Math Task Templates": https://ul.stanford.edu/sites/default/files/resource/2021-02/4.Language%20of%20Math%20Task%20Templates%2010-4-13.pdf

## *Resource #3:* PRINCIPLES FOR THE DESIGN OF MATHEMATICS CURRICULA

These principles and accompanying MLRs provide guidance to mathematics teachers for recognizing and supporting students' language development processes in the context of mathematical sensemaking. The document provides a framework for organizing strategies and special considerations to support students in learning mathematics practices, content, and language. See https://ul.stanford.edu/resource/principles-design-mathematics-curricula.

## *Resource #4:* ILLUSTRATIVE MATHEMATICS

A description of the MLRs and lessons in illustrative math courses that include MLRs—"The mathematical language routines … are effective and practical for simultaneously learning mathematical practices, content, and language. A mathematical language routine is a structured but adaptable format for amplifying, assessing, and developing students' language. The routines emphasize uses of language that [are] meaningful and purposeful, rather than about just getting answers. These routines can be adapted and incorporated across lessons in each unit to fit the mathematical work wherever there are productive opportunities to support students in using and improving their English and disciplinary language use." These routines facilitate attention to student language in ways that support in-the-moment teacher-, peer-, and self-assessment. The feedback enabled by these routines will help students revise and refine not only the way they organize and communicate their own ideas but also the way they ask questions to clarify their understandings of others' ideas. See https://im.kendallhunt.com/k5/teachers/teacher-guide/mathematical-language-development.html

## *Resource #5:* SUPPORTING CLASSROOM DISCUSSIONS

The following references are useful for supporting rich classroom discourse:

- The 5 Practices for Orchestrating Productive Mathematics Discussions are "anticipate," "monitor," "select," "sequence," and "connect" (Smith & Stein, 2011). In a problem-based curriculum, many activities can be described as "do math and talk about it," but the five practices lend more structure to these activities to help students make connections and learn new mathematics:
  - **Anticipate:** "Actively envision how students might approach the mathematics task they will work on."
  - **Monitor:** "[Pay] close attention to students' mathematical thinking and solution strategies as students work the task," individually or in small groups.
  - **Select:** "Select particular students to share their work with the rest of the class to get specific mathematics into the open for examination."
  - **Sequence:** "Make decisions regarding how to sequence the student presentation" to maximize connections between and among ideas.
  - **Connect:** "Help students draw connections between their solution and other students' solutions as well as the key mathematical ideas in the lesson." (Stein et al., 2008)
- *Classroom Discussions: Using Math Talk to Help Students Learn* (Chapin et al., 2009)
- *Promoting conceptual thinking in four upper-elementary mathematics classrooms* (Kazemi & Stipek, 2009)
- *Intentional Talk: How to Structure and Lead Productive Mathematical Discussions* (Kazemi & Hintz, 2014)
- *English language learners in the mathematics classroom* (Coggins et al., 2007)

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